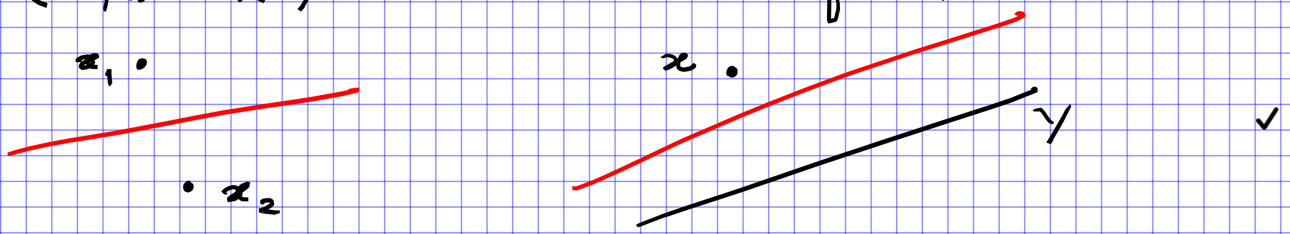
Functional Analysis I - L17 12/11/2020

Separation of Convex Sets & Extremality

① already seen

$(X, \|\cdot\|_X)$ be a normed vector space $/ \mathbb{R}$.



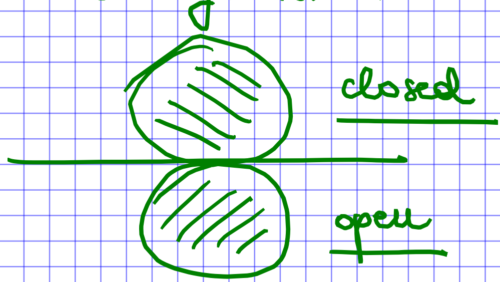
Thm (separation of convex sets) $A, B \subset X$ convex, disjoint (not empty).

i) if A is open, then $\exists\, \ell \in X^*$, $\lambda \in \mathbb{R}$ w/

$\forall a \in A, \forall b \in B$ $\boxed{\ell(a) < \lambda \leq \ell(b)}$

(weak separation property)

Take e.g. in $\mathbb{R}^2$ $A = B_1((0,-1))$ $B = \overline{B_1((0,1))}$



In this case

$$\begin{cases} \ell(x,y) = y & \text{(2nd coordinate)} \\ \lambda = 0 \end{cases}$$

ii) if A is compact and B is closed, then $\exists\, \ell \in X^*$, $\lambda \in \mathbb{R}$ w/

$$\sup_{a \in A} \ell(a) < \lambda < \inf_{b \in B} \ell(b)$$

(strong separation property)

The sets above do not satisfy the assumptions of part ii) and indeed they cannot be strongly separated.

Rmk. in both cases we call the ("a") separating hyperplane the affine hyperplane $H = \{x \in X : \ell(x) = \lambda\}$ ← λ - level set of ℓ

Key tool : Minkowski functional $\leftarrow$ "anisotropic norm"

Def: Let $C \subset X$ be a convex open set containing the origin. We call Minkowski functional of C, $p_C : X \longrightarrow \mathbb{R}$

$$p_C(x) = \inf\{\lambda > 0 : \underbrace{x \in \lambda C}\}.$$

Some properties:

(1) $B_R = C \rightsquigarrow p(x) = \dfrac{\|x\|}{R}$

ball of radius $R>0$ centered at the origin in X

Exercise: how about an ellipse $E(a,b)$ in $\mathbb{R}^2$?

(2) $C_1 \subset C_2 \rightsquigarrow p_{C_2}(x) \leq p_{C_1}(x) \quad (\forall x \in X)$

(3) **Lemma**: the Minkowski functional (of a convex set, containing 0) is a sublinear functional, i.e.

$$(1) \quad p(x+y) \leq p(x) + p(y) \quad (\forall x, y \in X)$$

$$(2) \quad p(\lambda x) = \lambda p(x) \quad \forall \lambda \geq 0 \quad (\forall x \in X)$$

Pf. (2) is a validity check. Let's assume it and do (1).

Assume we can write $x = \lambda u$ for some $\lambda > 0$, $u \in C$ (a representation of x as an element of λC)

Similarly $y = \mu v$ for some $\mu > 0$, $v \in C$ (...)

Set $z = \dfrac{\lambda u + \mu v}{\lambda + \mu} = \left(\dfrac{\lambda}{\lambda+\mu}\right) u + \left(\dfrac{\mu}{\lambda+\mu}\right) v$

convex combination of $u, v \in C \implies z \in C$

$\implies p(z) \leq 1$

$$p(x+y) = p(\lambda u + \mu v) = (\lambda+\mu)\, p(z) \leq \lambda + \mu$$

for any such representation using (2)

this inequality is now true $\forall\ \lambda, \mu > 0$ as above (...)

$$\Rightarrow \quad p(x+y) \le \inf_{\substack{\lambda > 0 \\ x = \lambda u,\ \text{for } u \in C}} \ \inf_{\substack{\mu > 0 \\ y = \mu v,\ \text{for } v \in C}} (\lambda + \mu)$$

$$= \underbrace{\inf_{\substack{\lambda > 0 \\ (\ \text{———}\ )}} \lambda}_{p(x)} + \underbrace{\inf_{\substack{\mu > 0 \\ (\ \text{—}\ )}} \mu}_{p(y)}$$

■

Pf (separation thm, part i)):

fix $a_0 \in A$, $b_0 \in B$ set $x_0 := b_0 - a_0$.

Define $C := A - B + x_0 = \{a - b + x_0 \mid a \in A, b \in B\}$

check:
- $C \neq \emptyset$
- $0 \in C$
- C open (the sum/difference of two sets is open as soon as one of the sets is ...)
- C convex
- $x_0 \notin C$ (why? because by hp. $A \cap B = \emptyset$)

Fact 1: from above $\exists\ R > 0$ s.t. $B_R(\underline{0}) \subset C$

So $p(x) \le M \|x\|_X \quad \forall x \in X \quad M = 1/R$

Fact 2: since C is open $C = \{p(x) < 1\}$

(check two inclusions: $\{p < 1\} \subset C$ is always true, other incl. $C \subset \{p < 1\}$ needs C to be open)

Fact 3: pick $x_0 = b_0 - a_0$ as above.

$$x_0 \notin C \underset{\text{Fact 2}}{\Longrightarrow} \boxed{p(x_0) \ge 1}$$

So if we let $f: \operatorname{span}\{x_0\} \longrightarrow \mathbb{R}$

$$t x_0 \longmapsto t$$

then f is dominated by p, since

$$t \geq 0 \qquad f(tx_0) = t \leq t\,p(x_0) \overset{\text{homog.}}{=} p(tx_0)$$

$$t < 0 \qquad f(tx_0) = t < 0 \leq p(tx_0)$$

thus by HB $\exists$ linear extension $\ell : X \to \mathbb{R}$ of f

$$\ell(tx_0) = t\,\ell(x_0) \quad \forall t \qquad \text{and} \qquad \boxed{\ell(x) \leq p(x) \quad \forall x}$$

Fact 4: by fact $\underline{3}$ & fact 1 ($\hookrightarrow p(x) \leq \frac{\|x\|}{R}$)

have $\boxed{\ell \in X^*}$

Fact 5: given any $a \in A$, $b \in B$

$$\ell(a) - \ell(b) = \ell(a-b)$$

$$= \underbrace{\ell(a-b+x_0)}_{\substack{\leq p(a-b+x_0) \\ < 1}} - \underbrace{\ell(x_0)}_{=1} < 0$$

(fact 2)

So $\ell(a) < \ell(b)$ $\forall a \in A, \forall b \in B$ $\Rightarrow$ $\underbrace{\sup_{a \in A} \ell(a) \leq \lambda = \inf_{b \in B}}\ell(b)$

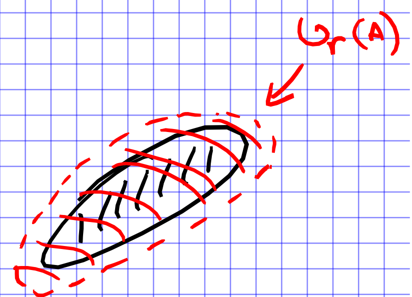
SEE ADDENDUM at the end of this lecture (#)

Now: prove part ii).

Topological rmk: if A compact, B closed then $\exists\, r > 0$ s.t. (advice: argue by contradiction...)

$$U_r(A) \cap B = \emptyset$$

where $U_r(A) := \bigcup_{a \in A} B_r(a)$



$\Downarrow$

we apply part i) to $A' = U_r(A)$ and B

A': open; B: given closed set

$$\sup_{a' \in A'} \ell(a) \leq \inf_{b \in B} \ell(b) \qquad (1)$$

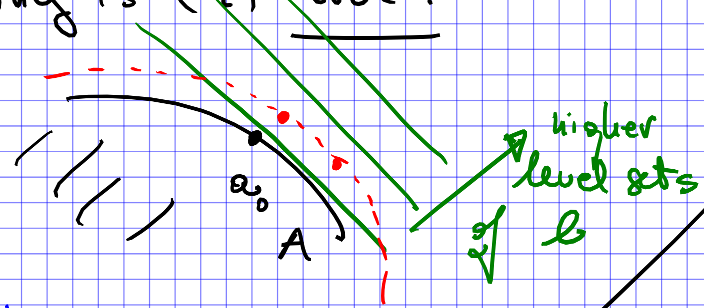
But on the other hand, I claim:

$$\underbrace{\max_{a \in A} \ell(a)}_{\text{existence by Weierstrass}} < \sup_{a' \in A'} \ell(a) \qquad (2)$$

thus (1) and (2) give

$$\max_{a \in A} \ell(a) < \inf_{b \in B} \ell(b)$$

Why is (2) true?



if (2) were not true, then the opposite

$$\sup_{a' \in A'} \ell(a) \leq \max_{a \in A} \ell(a)$$

in part. call $a_0 \in A$ a maximizer for

$$\max_{a \in A} \ell(a)$$

(*) thus $\ell(a_0 + tv) \leq \ell(a_0)$ $\boxed{\forall v \in B_{\epsilon = r/2}(0)}$ $\forall t \in (-1, 1)$

So the function $t \longmapsto \ell(a_0 + tv)$ has a local max at $t = 0$

$\overset{\curvearrowright}{\text{1st der test}}$. $\underbrace{\frac{d}{dt}\Big|_{t=0} \ell(a_0 + tv)}_{\equiv \ \ell(v)} = 0 \implies \ell \equiv 0$ in X^* impossible by construction.

Proved: ineq. (2) is true, thus (ii) of the proof is also complete ∎

ADDENDUM (#) for part i): we proved $^{(\#_1)}$ $\ell(a) \leq \lambda := \inf_{b \in B} \ell(b)$

but we had claimed the stronger output that$^{(\#_2)}$ $\ell(a) < \lambda = \inf_{b \in B} \ell(b)$ $\forall a \in A$

If $(\#_2)$ were false then $\exists\, a_0 \in A$ with $\ell(a_0) = \lambda$. Such point is a maximum for the problem $\max_{a \in A} \ell(a)$, but A is open so I can argue as above to derive that this would force $\ell \equiv 0$ (the zero functional) which is impossible since by construction $\ell(a) < \ell(b)$ $\forall a \in A$ $\forall b \in B$.

Extremal points:

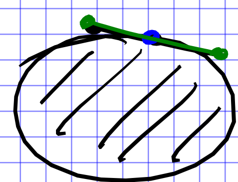
$(X, \|\cdot\|_X)$ normed space as above, $/\mathbb{R}$. $K \subset X$ subset.

Def: A subset $M \subset K$ will be called extremal for K (also: an extremal set for K) if

$$\left.\begin{array}{l} x_\alpha = \alpha x_1 + (1-\alpha) x_0 \in M \\ 0 < \alpha < 1 \quad x_0, x_1 \in K \end{array}\right. \implies x_0, x_1 \in M$$

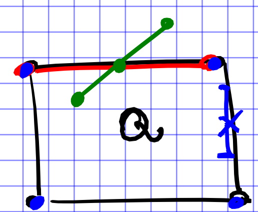
A point $x \in K$ is called extremal if $M = \{x\}$ is an extremal subset for K.

Example: $\cdot\ \mathbb{R}^2$



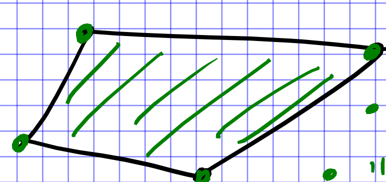
$K = \overline{B}$

- any point of $S^1 = \partial B$ is extremal (and only those)



- any side e.g. $[-1,1] \times \{1\}$ is an extremal set for Q
- the extremal points are just the 4 vertices.

Goal (Krein-Milman)



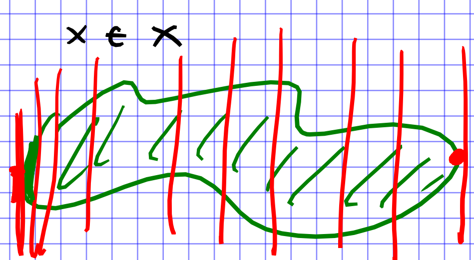
- $\exists$ extremal points
- "reconstruction"

Lemma 1 (transitivity) Setting as above:

$$\left.\begin{array}{l} M \subset K \text{ an extremal subset of } K \\ L \subset M \text{ an extremal subset of } M \end{array}\right\} \implies L \subset K \text{ is an extremal subset of } K$$

Lemma 2 $K \subset X$ compact, any $\ell \in X^*$.

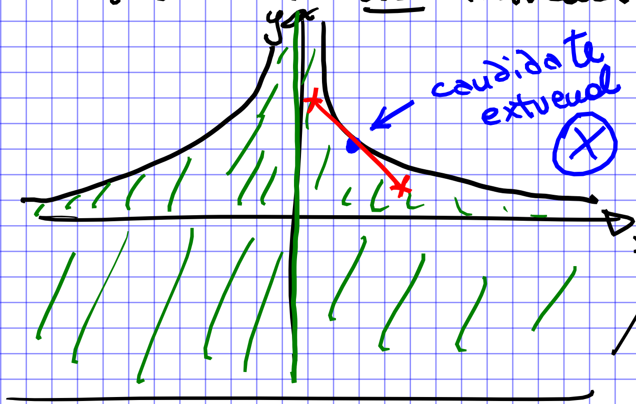
$\lambda = \min_{x \in X} \ell(x)$ let $K_\lambda = \{x \in X : \ell(x) = \lambda\}$ is an extremal set for K



Thm (Krein - Milman) Let $K \neq \emptyset$ be compact (but not nec. convex) Then K has extremal points. (Existence)

Rmk. (compactness is needed!)

a set with no extremal points in $\mathbb{R}^2$ is e.g.



$$M = \{ (x,y) \in \mathbb{R}^2 : \quad y \leq \frac{1}{|x|} \}$$

Pf. (apply Zorn's Lemma)

$\mathcal{M} := \{ M \subset K, \ M \neq \emptyset, \text{ compact and extremal} \}$

check: $\mathcal{M} \neq \emptyset$ because $K \in \mathcal{M}$ (any set is tautologically extremal for itself...)

> partial order on $\mathcal{M}$:

$$\forall L, M \in \mathcal{M} : \quad L \leq M \Longleftrightarrow M \subseteq L$$

check: $(\mathcal{M}, \leq)$ satisfies the chain condition

$\mathcal{M}_\iota$ be a $\underbrace{\text{totally ordered subclass of}}_{\text{chain}}$ $(\mathcal{M}, \leq)$

then there are upper bounds

set $M := \bigcap_\iota M_\iota$

claim: (i) $M \neq \emptyset$

(ii) M compact

(iii) M extremal for K

if so, chain condition is true

check claim iii)

$$x_\alpha = \alpha x_1 + (1-\alpha) x_0$$

$$\in M \rightsquigarrow x_\alpha \in M_i \quad \forall \iota$$

but M_i is extremal in K

$\Longrightarrow x_0, x_1 \in M_i \quad (\forall i)$

$\Longrightarrow x_0, x_1 \in M$

<u>check claim i) and ii)</u> recall FIP (i.e. FIP characterization of compactness)

(X, τ) top. space is compact $\Longleftrightarrow$ every collection of closed subsets w/ FIP (i.e. any intersection of finitely many closed subsets is not empty) has non-empty intersection

complementing game

<u>Application above</u>: K compact so $\uparrow$ holds. In part. the family M_i, which patently has the FIP

(because $M_{i_1} \cap \dots \cap M_{i_k} = M_{i_k}$ if (wlog) $M_{i_1} \supseteq M_{i_2} \supseteq \dots \supseteq M_{i_k}$)

thus $M = \bigcap M_i$ is <u>not</u> empty. Also M is closed,

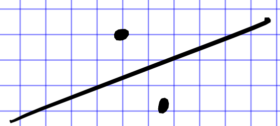
$M \subset K \Longrightarrow M$ compact
$\uparrow$ compact

<u>Zorn</u> $\Longrightarrow \exists\, M \in \mathcal{M}$ maximal w.r.t. $\leq$

$(M \in \mathcal{M} \Rightarrow M \neq \phi)$. Pick $x \in M$, claim $\boxed{M = \{x\}}$

by contradiction, assume not so let $y \in M$ $y \neq x$

$\exists\, \ell \in X^*$ separating



$\lambda = \min_{m \in M} \ell(m)$ (Weierstrass!)

$M_\lambda = \{ m \in M , \ \ell(m) = \lambda \} \subset M$

- M_λ extremal in M (by Lemma 2)
- hence (since M extremal in K), by Lemma 1 M_λ extremal in K

So $M_\lambda \in \mathcal{C}$, also $M_\lambda \subsetneq M$ (because only one among x and y can be in M_λ)

$\Rightarrow M$ is not maximal. $\lightning$ ∎

<u>Def</u> Setting as above, let $A \subset X$ we define the <u>closed convex hull</u> of A as $\overline{\text{conv}}(A) = \bigcap_{\substack{A \subset B \\ B \text{ convex, closed}}} B$.

<u>Thm</u> (Krein-Milman II, reconstruction)

Let K be compact and convex, $E \subset K$ be the set of its extremal points. Then $\boxed{K = \overline{\text{conv}}(E)}$

<u>Pf</u>. K compact in $(X, \|\cdot\|_X)$ [that is T_2, being a metric space] is then <u>closed</u>, and thus $K \supseteq \overline{\text{conv}}(E)$. By contradiction, let $K \supsetneq \overline{\text{conv}}(E)$, thus pick $x_0 \in K \setminus \overline{\text{conv}}(E)$. By the <u>strong</u> separation thm. applied to $A = \{x_0\}$ and $B = \overline{\text{conv}}(E)$ there is $\ell \in X^*$ w/ $\inf_{x \in \overline{\text{conv}}(E)} \ell(x) > \ell(x_0) \geq \min_{x \in K} \ell(x) =: \lambda$

Have: $K_\lambda \neq \emptyset$, compact (since it is closed in compact) and extremal in K (by Lemma 2).

By Krein-Milman I $\exists$ y_0 extremal point for K_λ, thus by Lemma 1 (transitivity) have that y_0 is extremal in K as well.

So $y_0 \in E \Rightarrow \ell(y_0) > \lambda$, a contradiction since $y_0 \in K_\lambda$ $\lightning$