Functional Analysis I – L18 16/11/2020

Weak convergence and weak topology ✓

① Motivation: the direct method of the Calculus of Variations

setup $(X, \|\cdot\|)$ Banach space

$\overline{B} = \{ x \in X : \|x\| \leq 1 \}$ closed unit ball

$f : X \longrightarrow \mathbb{R}$ at least C^0, "as nice as you want"

Problem

$$\inf_{x \in \overline{B}} f(x) = \lambda \in [-\infty, +\infty)$$

- $\lambda \in \mathbb{R}$?
- is there a minimiser ? i.e. does there exist $x_0 \in \overline{B}$ w/

$$f(x_0) = \min_{x \in \overline{B}} f(x) \quad ?$$

rmk. similarly one can ask e.g.

$$\inf_{x \in X} f(x) \qquad \text{provided say} \quad \lim_{\|x\| \to \infty} f(x) = +\infty$$

Direct Method

a) take a minimising seq. i.e. (x_k) in $\overline{B}$ w/

$$f(x_k) \longrightarrow \lambda$$

$\overline{B}$ never compact if $\dim X = \infty$

b) would like to exploit $\boxed{\text{sequential compactness of } \overline{B}}$ to extract a converging subsequence i.e. $x_{k(n)} \xrightarrow[n \to \infty]{} \overline{x} \in \overline{B}$

c) would like to exploit continuity to argue that

$$f(x_{k(n)}) \longrightarrow f(\overline{x})$$
$$\qquad\qquad\qquad\qquad\quad \| $$
$$\qquad\qquad\qquad\qquad\quad \lambda$$

$\rightsquigarrow$ $\lambda \in \mathbb{R}$ and $\overline{x} \in \overline{B}$ is a minimiser

Moral: must take a detour, in part. define a different topology on X.

② Weak convergence

Setup : $(X, \|\cdot\|)$ normed space , X^* top. dual

$(x_k) \subset X \qquad x \in X$

Def. We say $x_k \xrightarrow[w]{} x$ (converges weakly to x)

if $\forall \ell \in X^*$ $\qquad \ell(x_k) \longrightarrow \ell(x)$.

- Fact 1 : (uniqueness of the limit) $\begin{cases} x_k \xrightarrow[w]{} x \\ x_k \xrightarrow[w]{} x' \end{cases}$ then $x = x'$.

why? by HB $\exists\ \ell \in X^*$ w/ $\ \ell(x) \neq \ell(x')$

but $\quad x_k \xrightarrow[w]{} x \implies \ell(x_k) \longrightarrow \ell(x)$

$\qquad x_k \xrightarrow[w]{} x' \implies \ell(x_k) \longrightarrow \ell(x')$ $\qquad$ contradiction

- Fact 2 : (strong conv. $\Rightarrow$ weak conv.)

$$x_k \xrightarrow{\|\cdot\|} x \implies x_k \xrightarrow[w]{} x$$

why? $\quad |\ell(x_k - x)| \leq \|\ell\|_{X^*} \underbrace{\|x_k - x\|_X}_{\to 0}$

Example . a sequence converging weakly but not strongly.

$\ell^2 \longleftarrow$ model for all separable Hilbert spaces

standard (monomial) Hilbert basis $\quad e_i = (0, \dots, 0, 1, 0, \dots 0)$
$\qquad$ i-th slot

then $\boxed{e_i \xrightarrow[i \to 0]{w} 0}$ $\quad$ why? $\ \ell \in (\ell^2)^*$, must prove

$\ell(e_i) \xrightarrow[i \to \infty]{} 0$. But by Riesz $\exists\, z$ w/

$$\forall x \in \ell^2 \qquad \ell(x) = (z, x)_{\ell^2}$$

$$\ell(e_i) = (z, e_i) = z_i \longrightarrow 0$$

z_i i-th component of the vector z

true by Bessel inequality

$$\sum |z_i|^2 = |z|^2 < \infty$$

· **Fact 3** (l.s.c. of $\|\cdot\|$ w.r.t. weak convergence)

$x_k \xrightarrow[w]{} x$. Then (x_k) is <u>bounded</u> in X

and $\|x\| \leq \liminf_{k\to\infty} \|x_k\|$.

<u>Pf</u>: Consider for $k \in \mathbb{N}$ $\quad A_k : X^* \longrightarrow \mathbb{R}$

$$\ell \longmapsto \ell(x_k)$$

i.e. $\quad A_k(\ell) = \ell(x_k)$

Now, X^* is <u>Banach</u> and A_k are <u>pointwise bounded</u>

$$\left(A_k(\ell) = \ell(x_k) \longrightarrow \ell(x)\right)$$

Banach-Steinhaus $\Longrightarrow$ $\{A_k\}$ is uniformly bounded

i.e. $\displaystyle\sup_{\|\ell\|_{X^*}\leq 1} |A_k(\ell)| = C < \infty$

$$\|x_k\|_X \underset{\substack{\uparrow \\ \text{dual characterization of} \\ \text{the norm}}}{=} \sup_{\|\ell\|_{X^*}\leq 1} |\ell(x_k)| < \infty$$

<u>Second part</u>: given $x \in X$ the limit point of (x_k)

pick $\ell \in X^*$ w/ $\|\ell\|_{X^*} = 1$ and $\ell(x) = \|x\|_X$.

$$\|x\|_X = \ell(x) \underset{\substack{\uparrow \\ \text{weak conv.}}}{=} \lim_{k\to\infty} \ell(x_k)$$

$$= \liminf_{k\to\infty} \underbrace{\ell(x_k)}_{|\cdot| \leq \|\ell\|_{X^*}\|x_k\|_X}$$

$$\leq \liminf_{k\to\infty} \underbrace{\|\ell\|_{X^*}}_{=1} \|x_k\|_X$$

$$= \liminf_{k\to\infty} \|x_k\|_X \qquad \blacksquare$$

③ From convergence to topology

Def: Given $(X, \|\cdot\|)$ as above, we call **weak topology** on X denoted τ_w the smallest topology such that the sets

$$\Omega_{\ell,U} = \ell^{-1}(U) \qquad \ell \in X^*, \ U \subset \mathbb{R} \text{ open}$$

are open. (assign a topology via a sub-basis)

Fact: equivalently an open set in τ_w is an arbitrary union of finite intersections of sets of the form $\Omega_{\ell,U}$, i.e. a **basis** for τ_w consists of all sets in X of the form

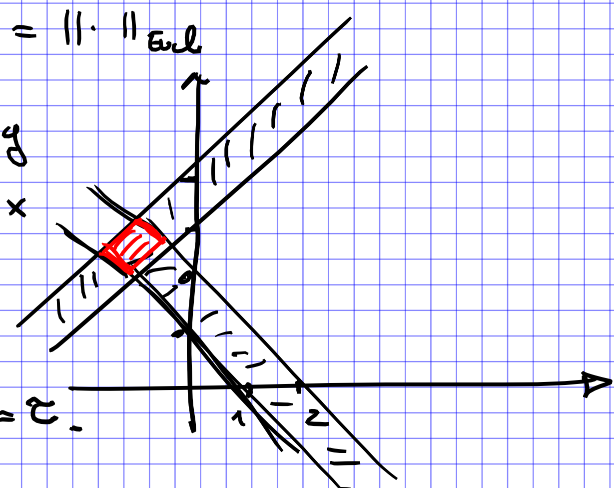
$$\Omega_{\ell_1,U_1} \cap \Omega_{\ell_2,U_2} \cap \dots \cap \Omega_{\ell_k,U_k} \qquad \ell_i \in X^*, \ U_i \subset \mathbb{R} \text{ open.}$$

Example/picture: $X = \mathbb{R}^2$, $\|\cdot\| = \|\cdot\|_{Eucl}$

$U_1 = (1,2)$ $\quad \ell_1(x,y) = x+y$

$U_2 = (3,4)$ $\quad \ell_2(x,y) = y-x$

draw $\Omega_{\ell_1,U_1} \cap \Omega_{\ell_2,U_2}$



$\hookrightarrow$ HW: if $\dim X < \infty$ then $\tau_w = \tau$.

Facts:

- $x_k \xrightarrow[w]{} x \iff x_k \to x$ w.r.t. τ_w
- (cf. uniqueness of the limit) τ_w is Hausdorff (T_2)
- $\tau_w \subset \tau$ (any open set in the weak topology is also open for the strong topology, but the converse is FALSE if $\dim X = \infty$)

equivalently: any closed set in the weak top. is also closed for the strong top., but the converse is FALSE if $\dim X = \infty$.

• [non-trivial] τ_w is not METRIZABLE in general
(guided project, see problem 10.1)

$\overset{\rightsquigarrow}{\text{implication}}$ one cannot say that it is equivalent for a set $\subset X$ to be closed in τ_w or to be $\underbrace{\text{sequentially closed}}$

$(x_k) \subset A$ and $x_k \to x$
$\Rightarrow x \in A$

Comparing closures :

Lemma : $(X, \|\cdot\|)$ normed space, $A \subset X$ subset. Then:

a) A closed $\Longleftrightarrow$ A seq. closed

b) A w-closed $\Longrightarrow$ A w-seq. closed

c) A w-closed $\Longrightarrow$ A closed $\quad (\tau_w \subset \tau)$

A w-seq. closed $\Longrightarrow$ A seq. closed. $\quad (\#)$

Pf. for a) and b) invoke a fact in general topology, i.e.

Fact : (X, τ) topological space, $A \subset X$ subset.
Then A closed $\Longrightarrow$ A seq. closed ;
the converse holds e.g. if (X, τ) is metrisable

Pf. $\boxed{\Rightarrow}$ given $A \subset X$ closed, take (x_n) in A, $x_n \to x$.
if by contradiction $x \notin A$ then $\mathcal{U} = X \setminus A$
open in X, $x \in \mathcal{U}$

but $(x_n) \subset A \Longrightarrow x_n \notin \mathcal{U} \;\forall n \in \mathbb{N}$
$\Longrightarrow x_n$ CANNOT converge to x.

$\boxed{\Leftarrow}$ let's prove that the complement of A is open.
Pick a point $x \notin A$: if $\exists\, k \in \mathbb{N}$ w/ $B_{1/k}(x) \cap A = \emptyset$
then we're done. Else $\forall k \in \mathbb{N}$ $\underbrace{B_{1/k}(x) \cap A}_{x_k} \neq \emptyset$ $\quad x_k \to x$
$\underset{A \text{ seq. closed}}{\Longrightarrow} x \in A$ ∎

(#) must check that $\left\{ \begin{array}{l} x_k \to x \\ (x_n) \subset A \end{array} \right. \Longrightarrow x \in A.$
given a w-seq. closed set A

but $[x_k \to x] \Longrightarrow [x_k \xrightarrow[w]{} x]$

$$\Longrightarrow x \in A \qquad \blacksquare$$

Example "the implication in c) is not an equivalence"

ℓ^2 $\quad S = \{ x \in \ell^2 : \|x\|_{\ell^2} = 1 \}$

- $S = \|\cdot\|^{-1}(\{1\})$ closed in $(X, \|\cdot\|_X)$

- S not weakly closed, and not w-seq. closed

$$e_i \xrightarrow[w]{} 0 \quad , \quad 0 \notin S$$

(proven above)

Example "the implication in b) is not an equivalence"

see Problem 9.6 or Struwe's lecture notes.

~ ∘ ~

One can rephrase the previous lemma as a comparison result for different notions of closure:

$(X, \|\cdot\|)$ normed, $\Omega \subset X$

$$\overline{\Omega}_w := \bigcap_{\substack{\Omega \subset A \\ A \ \tau_w\text{-closed}}} A$$

$(w\text{-clos}(\Omega))$

$$\overline{\Omega}_{w\text{-seq}} := \{ \text{limit points of sequences in } \Omega, \text{ w.r.t. convergence in } \tau_w \}$$

Lemma (rephrasing above)

a) b) c)

$$\begin{array}{ccc} \overline{\Omega} & \subset & \overline{\Omega}_w \\ \parallel & & \cup \\ \overline{\Omega}_{seq} & \subset & \overline{\Omega}_{w,seq}. \end{array}$$

c)

Theorem: Setting above, let $\Omega \subset X$ be a convex set (e.g. a subspace) Then $\overline{\Omega} = \overline{\Omega}_w \ \left(= \overline{\Omega}_{w,seq}\right)$