Functional Analysis I – L19   19/11/2020

Two key examples

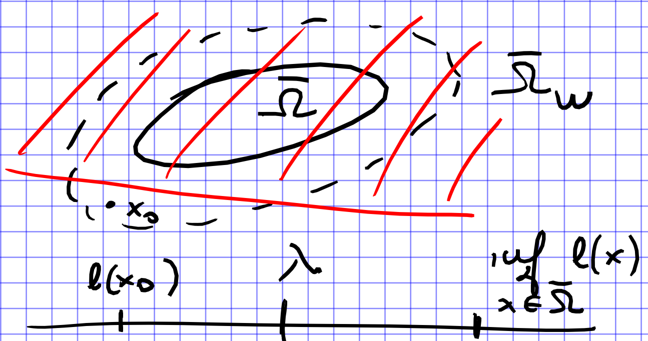
Thm Let $(X, \|\cdot\|)$ be a normed space ($/\mathbb{R}$) and let $\Omega \subset X$ be convex. Then $\overline{\Omega} = \overline{\Omega}_w = \overline{\Omega}_{w,seq}$.

Special cases:
- $\Omega = B_r(x)$ $x \in X$ $r > 0$
- $\Omega = \Pi$ affine hyperplane

Achtung: conclusions above are very much false for non-convex sets. ($\hookrightarrow$ example below)

Pf. recall general inclusions $\overline{\Omega} \subset \overline{\Omega}_{w,seq} \subset \overline{\Omega}_w$

By contradiction, assume $x_0 \in \overline{\Omega}_w \setminus \overline{\Omega} \neq \emptyset$.



$\overline{\Omega}$ is convex, so (strict separation thm)

$\exists\ \ell \in X^*$ w/

$$\ell(x_0) < \inf_{x \in \overline{\Omega}} \ell(x)$$

Now, if we define $K = \{x \in X : \ell(x) \geq \lambda\}$ ← closed halfspace

K is a closed set in the weak topology, which contains $\overline{\Omega}$ but not x_0. $\Rightarrow x_0 \notin \overline{\Omega}_w$ CONTRADICTION ∎

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For the next example, recall:

(X, τ) top. space, $A \subset X$ subset

$$\bar{a} \in \overline{A} \iff \forall\ \mathcal{U} \text{ open set},\ \bar{a} \in \mathcal{U}$$
$$\text{have } \mathcal{U} \cap A \neq \emptyset$$

We employ ↑ to study the next example.

Example   assume $\dim X = \infty$, and consider the unit sphere

$$S = \{x \in X : \|x\| = 1\}$$

- S is closed in $(X, \|\cdot\|)$, $S = \|\cdot\|^{-1}\{1\}$
- $\bar{S}_w = \bar{S}_{w,seq} = \underbrace{\{x \in X : \|x\| \leq 1\}}_{\bar{B}}$

why? $\bar{S}_w \subset \bar{B} \equiv \bar{B}_w$ (by previous thm)

Interesting part: given any $x_0 \in B$ have $x_0 \in \bar{S}_w$.

It's enough to prove that any

$$\underbrace{\Omega_{\ell_1,u_1} \cap \dots \cap \Omega_{\ell_n,u_n}}_{=:\Omega}$$

these sets form a basis for τ_w

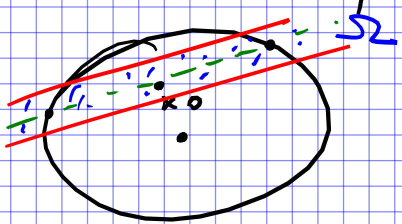
containing x_0, does have non-empty intersection w/ S.

FACT: given any $\ell_1, \dots, \ell_n \in X^*$ have that

set $K := \ker(\ell_1) \cap \dots \cap \ker(\ell_n)$

then $\forall v \in K$ $\underbrace{\{x_0 + tv : t \in \mathbb{R}\}}_{L_v}$ is contained

in $\Omega_{\ell_1,u_1} \cap \dots \cap \Omega_{\ell_n,u_n}$



FACT 2: $K \neq 0$ (here we use $\dim X = \infty$)

Given $\ell_1, \dots, \ell_n \in X^*$ we can build

$$\bar{\ell} : X \longrightarrow \mathbb{R}^n \quad \text{linear}$$

$$\bar{\ell} = (\ell_1, \ell_2, \dots, \ell_n)$$

if $K = 0$ then $\bar{\ell} : X \to \mathbb{R}^n$ injective. But a linear injective map sends lin. independent families of vectors to lin. indep. family of vectors.

Reflexive Spaces

$(X, \|\cdot\|_X)$ normed space

$$X^* := L(X, \mathbb{R}) \quad \text{w/ operator norm} \quad \|\ell\|_{X^*} = \sup_{\|x\|_X \le 1} |\ell(x)|$$

$$X^{**} := L(X^*, \mathbb{R}) \quad \text{w/ operator norm} \quad \|x^{**}\|_{X^{**}} = \sup_{\|\ell\|_{X^*} \le 1} |x^{**}(\ell)|$$

There is a canonical linear map:

$$\mathcal{I} : X \longrightarrow X^{**}$$
$$x \longmapsto \mathcal{I}x$$

for any fixed x $\quad \mathcal{I}x : X^* \longrightarrow \mathbb{R}$, $\quad \ell \longmapsto \ell(x)$ $\quad \boxed{\mathcal{I}x(\ell) = \ell(x)}$

<u>Prop.</u> $\mathcal{I}$ is a linear <u>isometry</u>. (i.e. it embeds X into X^{**} preserving distances)

<u>Pf.</u>

$$\|x\|_X \overset{\text{dual char. of norm}}{=} \sup_{\substack{\ell \in X^* \\ \|\ell\|_{X^*} \le 1}} |\ell(x)| = \sup_{\substack{\ell \in X^* \\ \|\ell\|_{X^*} \le 1}} |\mathcal{I}x(\ell)| \underset{\text{operator norm}}{=} \|\mathcal{I}x\|_{X^{**}}$$

∎

<u>Def</u>: the starting normed space $(X, \|\cdot\|_X)$ is called REFLEXIVE if $\mathcal{I} : X \longrightarrow X^{**}$ is <u>surjective</u>.

<u>Examples</u>:

① $\dim X < \infty \implies X$ reflexive

(dimension counting)

$$\dim X^{**} = \dim X^* = \dim X$$

$$\mathcal{I} \text{ injective} \implies \mathcal{I} \text{ surjective}.$$

② any Hilbert space is reflexive.

$$\begin{tikzcd} H \arrow[r, "J"] \arrow[rr, bend right, dashed, "\mathcal{I}"'] & H^* \arrow[r, "J^*"] & H^{**} \end{tikzcd}$$

$$\boxed{(Jx)(y) = (x, y)_H}$$ surjective by Riesz.

- Using J I can define a Hilbert space structure on H^* w/
$$(\ell, \ell')_{H^*} := (x, x')_H$$
where $\ell = Jx$, $\ell' = Jx'$

then I just mimic the construction above, i.e.

$$\boxed{(J^*\ell)(\ell') = (\ell, \ell')_{H^*}}$$ check $\boxed{\mathcal{I} = J^* \circ J}$

Since J, J^* are surjective then $\mathcal{I}$ surjective as well.

For the next to examples we need to recall/anticipate the notion of adjoint map:

$$\begin{tikzcd} X \arrow[r, "A"] \arrow[d, dash, "*"] & Y \arrow[d, dash, "*"] \\ X^* \arrow[d, dash, "*"] & Y^* \arrow[l, "A^*"'] \arrow[d, dash, "*"] \\ X^{**} \arrow[r, "A^{**}"] & Y^{**} \end{tikzcd}$$

$A \in L(X, Y)$

$A^* \in L(Y^*, X^*)$

- definition of A^*: $A^*\ell := \ell \circ A$ $(\ell \in Y^*)$
- $\| A \|_{L(X,Y)} = \| A^* \|_{L(Y^*, X^*)}$
- if A is a linear homeomorphism, then so is A^* i.e. $A^* : Y^* \to X^*$ is also a linear homeomorphism.

③ $1 < p < \infty$ $\quad L^p(\Omega)$ is <u>reflexive</u>.

$$\begin{tikzcd} L^p \arrow[r, "J^q"] \arrow[dr, dashed, "\mathcal{I}"'] & (L^q)^* \arrow[d, "((J^p)^*)^{-1}"] \\ & (L^p)^{**} \end{tikzcd}$$

$$\begin{tikzcd} L^q \arrow[r, "J^p"] \arrow[d, dash, "*"] & (L^p)^* \arrow[d, dash, "*"] \\ (L^q)^* & (L^p)^{**} \arrow[l, "(J^p)^*"'] \end{tikzcd}$$

<u>check</u>: $\mathcal{I} = (J^p)^{*^{-1}} \circ J^q$

$\Rightarrow \mathcal{I}$ <u>surjective</u>.

④ $p = 1$ $\quad L^1(\Omega)$ <u>not</u> reflexive.

$$\begin{tikzcd} L^1 \arrow[r, dashed, "J^\infty"] \arrow[dr, "\mathcal{I}"'] & (L^\infty)^* \\ & (L^1)^{**} \arrow[u, "(J^1)^*"'] \end{tikzcd}$$

if L^1 were reflexive, then (by def.) $\mathcal{I}$ would be surjective and so $J^\infty : L^1 \longrightarrow (L^\infty)^*$ is also surjective

BUT in L16 we prove it is <u>not</u>!

_q. is L^∞ reflexive or not!

<u>Facts</u>:

1) if X is reflexive, then it is complete (i.e. Banach)
(pf. X^{**} is <u>always</u> complete, but X reflexive implies $X \underset{homeo}{\cong} X^{**} \rightsquigarrow X$ <u>complete</u>)

2) (canonical completion) $\qquad$ <u>complete</u> (isometrically containing X)

$$X \longrightarrow \overline{\mathcal{I}(X)} \subset X^{**}$$

Prop. i) if X is reflexive, then X^* is reflexive,

ii) if X^* is reflexive and X is complete, then X is reflexive.

Key corollary: L^∞ is not reflexive (reduce to 4 done)

Proof (prop)

$$\mathcal{I} : X \longrightarrow X^{**}$$
$$\mathcal{I}^* : X^* \longrightarrow X^{***} \qquad \left(\text{not the adjoint of } \mathcal{I}\right)$$

For part i): we show that one can always solve

$$\mathcal{I}^* \ell = \underbrace{\ell^{**}}_{\text{given in } X^{***}} \qquad (\#)$$

both LHS and RHS define maps $X^{**} \longrightarrow \mathbb{R}$, so let's evaluate on the image of $\mathcal{I}$, i.e. on $\mathcal{I}X \subseteq X^{**}$, so

(necessary condition) $$\mathcal{I}^*\ell\,(\mathcal{I}x) = \ell^{**}(\mathcal{I}x)$$
$$\mathcal{I}x\,(\ell) \equiv \ell(x)$$

now we can use $\ell(x) := \ell^{**}(\mathcal{I}x)$

to define ℓ. Then $\uparrow$ this definition satisfies (#).

For part ii) assume (by contradiction) that $\underbrace{\mathcal{I}X}_{M} \subsetneq X^{**}$

M complete (since X is assumed to be complete!)

$\Longrightarrow$ M closed, linear subspace of X^{**}

pick $x^{**} \in X^{**} \setminus \mathcal{I}X$, so

$\exists\ \ell^{**} \in (X^{**})^* \equiv X^{***}$ w/

$\ell^{**}(x^{**}) = 1$, $\boxed{\ell^{**}|_M = 0}$

but we're assuming that $\mathcal{I}^*$ surjective, so in fact

$\exists\ \ell \in X^*$ w/ $\boxed{\ell^{**} = \mathcal{I}^* \ell}$

Thus: $\ell^{**}(\mathcal{I}x) = 0 \quad \forall x \in X \quad \Longrightarrow \quad \boxed{\ell \equiv 0 \text{ in } X^*}$

$$\mathcal{I}^* \ell\,(\mathcal{I}x) \underset{\text{def of } \mathcal{I}^*}{\equiv} \mathcal{I}x\,(\ell) \underset{\text{def of } \mathcal{I}}{\equiv} \ell(x)$$

$\Longrightarrow \ell^{**} = 0$ but $\ell^{**}(x^{**}) = 1$ ↯

Prop. $\left.\begin{array}{l} X \text{ reflexive} \\ Y \subset X \text{ closed linear subspace} \end{array}\right\} \Longrightarrow Y$ reflexive.

Pf. • $i_Y : Y \longrightarrow X$ (inclusion map) $\|i_Y\|_{L(Y,X)} = 1$

• $i_Y^* : X^* \longrightarrow Y^*$ $\qquad \|i_Y^*\|_{L(X^*,Y^*)} = 1$

$$i_Y^*(\ell)(y) = \ell(i_Y(y)) = \ell(y)$$

• $i_Y^{**} : Y^{**} \longrightarrow X^{**}$ $\qquad \|i_Y^{**}\|_{L(Y^{**},X^{**})} = 1$

$$i_Y^{**}(y^{**})(\ell) = y^{**}(i_Y^*(\ell))$$

Now: $\mathcal{I} : X \longrightarrow X^{**}$ assumed to be reflexive

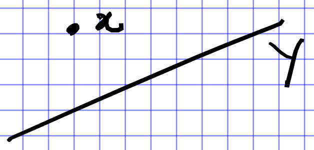
$\mathcal{I}^Y : Y \longrightarrow Y^{**}$ to be proven to be surjective

Natural Ansatz: given any $y^{**} \in Y^{**}$ the natural "candidate preimage" is $\boxed{\mathcal{I}^{-1}(i_Y^{**}(y^{**})) = y \in Y}$

This works but we first need to check

Claim: $\mathcal{I}^{-1}(i_Y^{**}(Y^{**})) \subset Y$

Pf. (claim) by contradiction, $x = \mathcal{I}^{-1}(i_Y^{**}(y^{**})) \notin Y$ (for some $y^{**} \in Y^{**}$)



pick $\ell \in X^*$

$\ell|_Y = 0 \iff \boxed{i_Y^*(\ell) = 0}$

$\ell(x) = 1$

$$0 = y^{**}(\underbrace{i_Y^*(\ell)}_{=0}) \overset{\uparrow}{=} (i_Y^{**}(y^{**}))(\ell) \overset{\uparrow}{=} (\mathcal{I}x)(\ell) = \ell(x)$$

equation defining i_Y^{**}

definition of x

Now check $y^{**} = \mathcal{I}^Y y$ for y defined in $\square$ above.

Given $f \in Y^*$ let $\ell \in X^*$ be an isometric extension of f (in part. $\ell|_Y = f \iff i_Y^*(\ell) = f$)

$$y^{**}(f) = y^{**}(i_Y^*(\ell)) = i_Y^{**}(y^{**})(\ell) \overset{\uparrow}{=} \mathcal{I}y(\ell) = \ell(y) = f(y) = \mathcal{I}^Y y(f)$$

definition of y

so: $\forall f \in Y^*$ we have proven

$$y^{**}(f) = \mathcal{I}^Y y(f)$$

$\Longrightarrow$ $y^{**} = \mathcal{I}^Y y$ so $\mathcal{I}^Y$ is surjective. ⊠

Rmk. last step of pf we actually showed

$$i_Y^{**} \mathcal{I}^Y = \mathcal{I}|_Y$$