Functional Analysis I - L20 23/11/2020

Separability

(M, d) metric space

Def: . A subset $D \subset M$ is called <u>dense</u> if $\overline{D} = M$.

. (M, d) is called separable if it has a <u>countable</u> dense subset.

<u>Examples:</u> 1) $\mathbb{R}^n$ w/ any metric induced by a norm $(D = \mathbb{Q}^n)$

2) $C^0([0,1])$ Weierstrass approx thm. $(D = \mathbb{Q}[x])$

3) $L^p(\Omega)$, $1 \leq p < \infty$ $(D = \{\text{simple functions}\})$

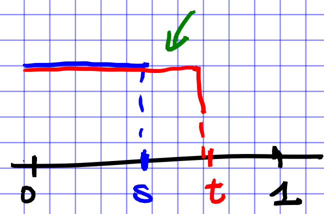
$\sum q_i \chi_{B_{r_i}}(x_i)$

$q_i \in \mathbb{Q}$, $r_i \in \mathbb{Q}_{>0}$, $x_i \in \mathbb{Q}^n$

4) $L^\infty(\Omega)$ ← NOT separable (cf. Problem 1.5)

$L^\infty([0,1])$ $\quad f_s := \chi_{[0,s]}$

$s \neq t \Rightarrow \|f_s - f_t\|_{L^\infty} = 1$



By contradiction, let (g_k) be dense in $L^\infty([0,1])$. In part. for any $0 < s \leq 1$ one can find <u>one</u> $k \in \mathbb{N}$ w/

$$\|f_s - g_k\|_{L^\infty} < 1/2 .$$

But for fixed k $\exists!$ s such that ↑ holds, for if not

$$\|f_s - f_{s'}\|_{L^\infty} \leq \|f_s - g_k\|_{L^\infty} + \|g_k - f_{s'}\|_{L^\infty} < 1 .$$

So $\exists \Lambda \subset \mathbb{N}$ and $\Lambda \xrightarrow{\mathcal{N}} (0,1]$ well-defined and <u>surjective</u>

<u>impossible!</u> ↯

<u>Two facts about separability:</u>

① <u>Thm.</u> Let (M, d) be separable, $A \subset M$ a subset. Then $(A, d|_{A \times A})$ is separable.

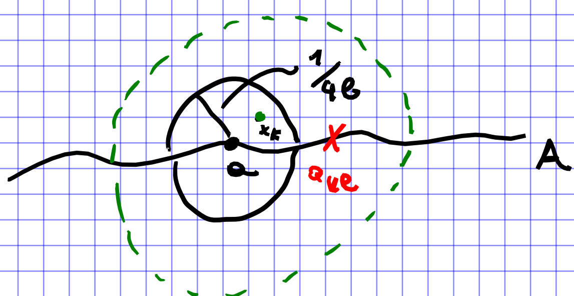
Pf. Let (x_k) be dense in (M,d).

Fix one $a_0 \in A$, then for $k \geq 0$, $\ell \geq 1$ $(k,\ell \in \mathbb{N})$

- if $B_{1/2\ell}(x_k) \cap A \neq \emptyset$ $\rightsquigarrow$ choose $a_{k\ell} \in B_{1/2\ell}(x_k) \cap A$
- if $B_{1/2\ell}(x_k) \cap A = \emptyset$ $\rightsquigarrow$ set $a_{k\ell} = a_0$

Claim: the family $(a_{k\ell})$ is dense in A.

Given $a \in A \subset M$ and $r > 0$ we must check the existence of some $a_{k\ell} \in B_r^A(a)$. So: choose $\ell > 0$ s.t. $1/\ell < r$, can find (by density of (x_k)) an element $x_k \in B_{1/4\ell}(a)$



$$\underbrace{B_{1/2\ell}(x_k) \cap A \neq \emptyset}_{\supset\, B_{1/4\ell}(a)} \qquad \{\text{triangle inequality}\}$$

we had chosen $a_{k\ell} \in B_{1/2\ell}(x_k) \cap A$

$$d(a_{k\ell}, a) < \text{diameter of given ball}$$
$$\qquad\qquad\qquad\qquad \shortparallel$$
$$2 \cdot 1/2\ell = 1/\ell < r \qquad \blacksquare$$

2 Separability and duality

Thm. Let $(X, \|\cdot\|_X)$ be a normed space $/\mathbb{R}$.

i) if X^* is separable, then so is X.

ii) if X is separable and reflexive, then X^* is separable.

Pf. i) Let (ℓ_k) be a sequence in X^*, dense.

For each $k \in \mathbb{N}$ can find (by def. of $\|\cdot\|_{X^*}$) $x_k \in X$

w/
$$\begin{cases} \ell_k(x_k) \geq \|\ell_k\|_{X^*} - 1/k & (\#) \\ \|x_k\|_X = 1 \end{cases}$$

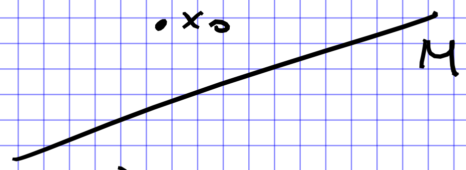
{pick the quasi-maxima for the separating family of functional}

Set $M := \overline{\text{span}\{x_n \,,\, n \in \mathbb{N}\}}$

$= \overline{\left\{ \sum_{\substack{I \subset \mathbb{N} \\ \text{finite}}} q_i x_i \,,\, q_i \in \mathbb{Q} \right\}}$

enough to check $M = X$. If not, pick $x_0 \in X \setminus M$

and $\ell \in X^*$ w/ $\begin{cases} \ell(x_0) = 1 \\ \ell|_M = 0 \end{cases}$



By density, consider a subsequence $(\ell_{n(k)})_k$

w/ $\ell_{n(k)} \xrightarrow[k\to\infty]{} \ell$ (norm convergence in X^*)

$$0 \neq \|\ell\|_{X^*} = \lim_{k\to\infty} \|\ell_{n(k)}\|_{X^*}$$

$$\overset{(*)}{\leq} \limsup_{k\to\infty} \ell_{n(k)}(x_{n(k)})$$

$$\leq \limsup_{k\to\infty} |(\ell_{n(k)} - \ell)(x_{n(k)})|$$

$\ell|_M = 0$

$\Rightarrow \ell(x_{n(k)}) = 0$

$$\leq \limsup_{k\to\infty} \underbrace{\|\ell_{n(k)} - \ell\|_{X^*}}_{\to 0} \underbrace{\|x_{n(k)}\|_X}_{= 1}$$

$$\leq 0$$

ii) reduction to i)

Since X is reflexive

$$\underset{\text{separable}}{X^{**}} \longleftarrow = \underbrace{\mathcal{I} X}_{\text{separable}} \qquad \xrightarrow{\text{apply part i)}} \quad X^* \text{ separable}$$

∎

Sequential Compactness Results

$(X, \|\cdot\|_X)$ normed space / $\mathbb{R}$, X^* dual X^{**} bidual

$\mathcal{I}: X \longrightarrow X^{**}$

Def: (weak* convergence on X^*) We'll say that a sequence (ℓ_k) in X^* w^*-converges to $\ell \in X^*$ if

$$\forall x \in X \qquad \ell_k(x) \longrightarrow \ell(x).$$

Rmk. 3 notions of convergence on X^*:

i) in norm, i.e. for $\|\cdot\|_{X^*}$ namely $\|\ell_k - \ell\|_{X^*} \to 0$

ii) weak convergence, as induced by $(X^*)^* \equiv X^{**}$

i.e. $\ell_k \xrightarrow{w} \ell$ if $\forall\, \omega \in X^{**}$ $\omega(\ell_k) \longrightarrow \omega(\ell)$

iii) weak* convergence, as defined above

i.e. $\ell_k \xrightarrow{w^*} \ell$ if $\forall x \in X$ $\ell_k(x) \longrightarrow \ell(x)$.

Now, if $\underbrace{\omega \in X^{**} \text{ belongs to } \mathcal{I}X}$ then

$$\omega(\ell_k) \longrightarrow \omega(\ell) \iff \ell_k(x) \longrightarrow \ell(x)$$

$\omega = \mathcal{I}x$ for some $x \in X$ $\qquad \mathcal{I}x(\ell) := \ell(x)$

$$[\ell_k \longrightarrow \ell] \implies [\ell_k \xrightarrow{w} \ell] \implies [\ell_k \xrightarrow{w^*} \ell]$$

If X is reflexive then

$$[\ell_k \xrightarrow{w} \ell] \iff [\ell_k \xrightarrow{w^*} \ell].$$

but in general they two notions of convergence are <u>not</u> equivalent.

Rmk. (the background topology)

The w^*-convergence is induced by τ_{w^*}-topology, that is the topology on X^* generated by

$$\Omega_{x,u} = \{\ell \in X^* : \ell(x) \in \mathcal{U}\}$$

where $x \in X$, $\mathcal{U} \subset \mathbb{R}$ open.

This is declared to be a sub-basis for the topology.

- Hierarchy of topologies $\tau_{w^*} \subset \tau_w \subset \tau$
- Again τ_{w^*} is <u>not</u> metrizable in general (cf. Problem 10.1)
- Inclusions: $\Omega \subset X^*$ $\qquad \overline{\Omega} \subset \overline{\Omega}_w \subset \overline{\Omega}_{w^*}$

<u>Theorem</u> (Banach–Alaoglu)

Let X be separable, $(\ell_k) \subset X^*$ be bounded

i.e. $\exists\, C > 0$ w/ $\|\ell_k\|_{X^*} \leq C \;\; \forall k$

Then $\exists\, \ell \in X^*$ s.t. ℓ_k sub-converges to ℓ w.r.t. τ_{w^*}, i.e. $\exists\, \underbrace{\Lambda \subset \mathbb{N}}_{\text{infinite subset}}$ for which $\lim\limits_{\substack{k\to\infty \\ k\in\Lambda}}^{w^*} \ell_k = \ell$.

{ the unit ball in (X^*, τ_{w^*}) is sequentially compact }

<u>Examples</u>:

1) $X = L^1(\Omega)$ separable, so can apply thm. above to $L^\infty(\Omega)$

i.e. given (f_n) bounded seq. in L^∞
there is $f \in L^\infty$
and a subsequence $(f_{n(k)})_k$ $\qquad f_{n(k)} \xrightarrow{w^*} f$

i.e. $\forall g \in L^1(\Omega)$ $\qquad \int_\Omega f_{n(k)}\, g \, d\mu \longrightarrow \int_\Omega f g \, d\mu$

Same story for $L^p(\Omega)$, $1 < p < \infty$.

2) Consider $X = L^\infty([0,1])$ <u>not</u> separable.
The conclusion of Banach-Alaoglu fails (in general) for its dual X^*.

For $0 < \epsilon \leq 1$ set $T_\epsilon : L^\infty([0,1]) \longrightarrow \mathbb{R}$

$$T_\epsilon f = \frac{1}{\epsilon} \int_0^\epsilon f(x)\, dx$$

<u>Claim 1</u> $\quad \| T_\epsilon \|_{X^*} = \sup_{\|f\|_{L^\infty} \le 1} |T_\epsilon f| \le 1$

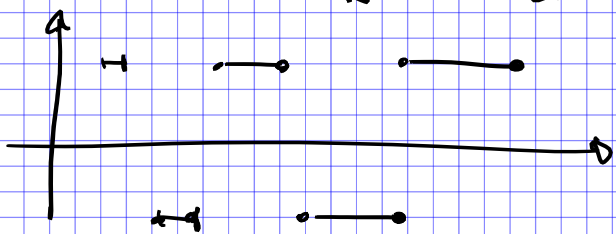
$\hookrightarrow$ we have a bounded family in $L^\infty([0,1])^*$

<u>Claim 2</u> the family $\{ T_\epsilon , \ 0 < \epsilon \le 1 \}$ is not weak* relatively subsequentially compact

By contradiction, assume instead $\exists \ (\epsilon_k) \quad \epsilon_k \searrow 0$

s.t. $\quad T_{\epsilon_k} \xrightarrow{w^*} T \in X^*$. wlog $\underbrace{\quad}_{\text{possibly excluding a subsequence}}$ $\dfrac{\epsilon_{k+1}}{\epsilon_k} \xrightarrow[k \to \infty]{} 0$

$\left(\text{e.g. } \dfrac{\epsilon_{k+1}}{\epsilon_k} < \dfrac{1}{2^k} \right)$ Pick: $\boxed{f := \sum_{k=1}^{\infty} (-1)^k \chi_{[\epsilon_{k+1}, \epsilon_k]}}$



If it were $T_{\epsilon_k} \xrightarrow{w^*} T$

in <u>part.</u> $\quad T_{\epsilon_k} f \longrightarrow Tf$

but this <u>cannot</u> possibly be the case.

$$T_{\epsilon_k} f = \frac{1}{\epsilon_k} \int_0^{\epsilon_k} f(x)\,dx = \frac{1}{\epsilon_k} \int_0^{\epsilon_{k+1}} f(x)\,dx + \frac{1}{\epsilon_k} \int_{\epsilon_{k+1}}^{\epsilon_k} f(x)\,dx$$

$$= \frac{1}{\epsilon_k} \int_0^{\epsilon_{k+1}} f(x)\,dx + (-1)^k \frac{\epsilon_k - \epsilon_{k+1}}{\epsilon_k}$$

$$= \frac{1}{\epsilon_k} \int_0^{\epsilon_{k+1}} f(x)\,dx + (-1)^k \left(1 - \frac{\epsilon_{k+1}}{\epsilon_k} \right)$$

$$\Rightarrow \left| T_{\epsilon_k} f - (-1)^k \right| \le \frac{\epsilon_{k+1}}{\epsilon_k} + \frac{1}{\epsilon_k} \epsilon_{k+1} = \frac{2\epsilon_{k+1}}{\epsilon_k} \xrightarrow[k \to \infty]{} 0$$

thus $a_k := T_{\epsilon_k} f$ is not Cauchy, because

$$|a_{k+1} - a_k| > 1 \quad \text{for } k \text{ large enough}$$

hence (a_k) cannot possibly converge. ∎

Pf. (Banach-Alaoglu)

Let $(x_j)_{j\in\mathbb{N}}$ be dense in $(X, \|\cdot\|_X)$. For any fixed $j\in\mathbb{N}$ the family $(\ell_k(x_j))_{k\in\mathbb{N}}$ is bounded in $\mathbb{R}$

so $\exists$ converging subsequence i.e. $\exists\ \Lambda_j\subset\mathbb{N}$ unbounded

w/ $\lim_{\substack{k\to\infty\\ k\in\Lambda_j}} \ell_k(x_j) = a_j \in \mathbb{R}$

[I can further require $\Lambda_1 \supset \Lambda_2 \supset \Lambda_3 \supset \dots$

Set $M := \operatorname{span}\{x_j : j\in\mathbb{N}\}$

define $\ell : M \to \mathbb{R}$ by $\ell(x_j) = a_j$

+ extension by linearity ($)

Idea: $\ell \in L(M,\mathbb{R})$

thus (by isometric extension lemma) ℓ extends to an element of $L(X,\mathbb{R}) = X^*$

We must still check that $\boxed{\ell_k \xrightarrow[\substack{k\to\infty\\ k\in\Lambda}]{w^*} \ell}$

Rmk. note that if Λ is a diagonal ($)

sequence extracted from the sets $\Lambda_1\supset\Lambda_2\supset\Lambda_3\supset\dots$

then in part. given $x\in M$ it holds $\ell(x) = \lim_{\substack{k\to\infty\\ k\in\Lambda}} \ell_k(x)$

Thus note that

$$|\ell(x)| = \lim_{\substack{k\to\infty\\ k\in\Lambda}} |\ell_k(x)| \le \underbrace{\left(\limsup_{k\to\infty} \|\ell_k\|_{X^*}\right)}_{\le C \text{ by hp.}} \|x\|$$

$$\Rightarrow \sup_{x\in M} \frac{|\ell(x)|}{\|x\|} \le C$$

and isometric extension is applicable!

Let's conclude by checking the claim above.

Fixed $x \in X$, by density (i.e. by the separability assumption) $\exists\ J \subset \mathbb{N}$ unbounded set of indices such that $\boxed{x = \lim_{\substack{j\to\infty \\ j \in J}} x_j}$.

Then:

$$|\ell_k(x) - \ell(x)| \le |\ell_k(x - x_j)| + |\ell_k(x_j) - \ell(x_j)| + |\ell(x_j - x)|$$

$$\le \left(\sup_k \|\ell_k\|_{X^*} + \|\ell\|_{X^*} \right) \|x_j - x\|_X + |\ell_k(x_j) - \ell(x_j)|$$

Hence, for any fixed $j \in J$, if we let $k \to \infty$ we get

$$\limsup_{\substack{k\to\infty \\ k \in \Lambda}} |\ell_k(x) - \ell(x)| \le 2C \|x_j - j\|_X$$

by ✓

so if we then let $j \to \infty$ $(j \in J)$ we must conclude that

$$\limsup_{\substack{k\to\infty \\ k \in \Lambda}} |\ell_k(x) - \ell(x)| = 0$$

i.e. $\ell_k(x) \to \ell(x)$, which completes the proof. ∎