Functional Analysis I – L21 26/11/2020

$\boxed{\text{Compactness thm for weak topology}}$

Thm. (Eberlein – Smulian)

Let $(X, \|\cdot\|)$ be a reflexive normed space. Given a bounded sequence (x_k), there exists $x \in X$ and $\Lambda \subset \mathbb{N}$ w/

$$x_k \xrightarrow{w} x \qquad (k \to \infty, k \in \Lambda).$$

{ for reflexive normed spaces, the unit ball is sequentially compact w.r.t. weak topology }

Examples of applicability:

i) Hilbert spaces

ii) $L^p(\Omega)$, $1 < p < \infty$

Pf. (reduction to applying Banach – Alaoglu)

$$Y = \overline{\operatorname{span}_{\mathbb{R}} \{x_k : k \in \mathbb{N}\}} \subset X \quad \text{closed}$$

- Y separable (approx. finite linear comb / $\mathbb{Q}$)
- Y reflexive ("closed subspace of reflexive is reflexive")

$\Rightarrow Y^*$ separable

Banach – Alaoglu (applied to Y^{**}) gives (possibly extracting a subsequence)

$$\mathcal{I} x_k \xrightarrow{w^*} \mathcal{I} x \quad \text{for some } x \in Y$$

i.e. $\boxed{\forall\, \ell \in Y^*}$

$$\begin{array}{ccc} (\mathcal{I} x_k)(\ell) & \xrightarrow{k \to \infty} & (\mathcal{I} x)(\ell) \\ \parallel\!\!| & & \parallel\!\!| \\ \ell(x_k) & \longrightarrow & \ell(x) \end{array}$$

This is true (by restriction) for all $\ell \in X^*$: because given $\boxed{\ell \in X^*}$

$\ell|_Y \in Y^*$

$$\begin{array}{ccc} \ell|_Y(x_k) & \longrightarrow & \ell|_Y(x) \\ \parallel\!\!| & & \parallel\!\!| \\ \ell(x_k) & \longrightarrow & \ell(x) \end{array}$$

i.e. $x_k \xrightarrow{w} x$ ∎

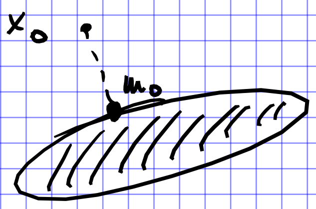
1^{st} application

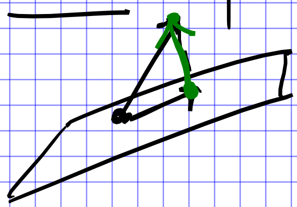
Let X be a reflexive normed space, $M \subset X \neq \emptyset$ closed and convex, $x_0 \in X \setminus M$.

Then there exist $m_0 \in M$ w/

minimum problem

$$\| x_0 - m_0 \| = \operatorname{dist}(x_0, M) := \boxed{\inf_{m \in M} \| x_0 - m \|_X}.$$





rmk. cf. "least distance points from a closed subspace in Hilbert"

cf. problem 7.6

Pf. (direct methods)

pick $(m_k) \subset M$ minimizing seq., i.e.

$$\operatorname{dist}(x_0, M) = \lim_{k \to \infty} \| x_0 - m_k \|_X$$

- (m_k) bounded : $\| m_k \|_X \leq \| x_0 \|_X + \underbrace{\| m_k - x_0 \|_X}_{\to \operatorname{dist}(x_0, M)}$

$$\xRightarrow{E-S} \quad \exists\, m_0 \in X \text{ w/ } m_k \xrightarrow{w} m_0 \quad (k \to \infty).$$

M convex closed (w.r.t. $\| \cdot \|_X$) $\Longrightarrow$ M weakly seq. closed $\Longrightarrow$ $m_0 \in M$

- by l.s.c. of $\| \cdot \|_X$ (see 1^{st} lecture on weak conv.)

$$\| m_0 - x_0 \|_X \leq \underbrace{\liminf_{k \to \infty} \| m_k - x_0 \|_X}_{= \operatorname{dist}(x_0, M)}$$

$m_0 \in M$ is the desired minimizer. ∎

Rmk. the minimum (point) may/may not be unique.

For instance, it is unique if $\|\cdot\|_X$ is strictly convex, in part. if $\left\| \frac{a+b}{2} \right\| < \frac{1}{2} \left(\|a\| + \|b\| \right)$

why? if w_0, w_0' were two minima then

$$\left\| x_0 - \frac{w_0 + w_0'}{2} \right\|_X = \frac{1}{2} \left\| (x_0 - w_0) + (x_0 - w_0') \right\|$$

$$\underset{\text{Strict convexity of norm}}{<} \frac{1}{2} \Big(\underbrace{\|x_0 - w_0\|}_{= d} + \underbrace{\|x_0 - w_0'\|}_{d} \Big)$$

$\Rightarrow \frac{w_0 + w_0'}{2} \in M$ has distance from x_0 less than $d(x_0, w_0)$ $\lightning$ ∎

Precise tool for Calculus (Mazur's Lemma, cf. problem 9.7)

Let $(x_k) \subset X$ w/ $x_k \xrightarrow{w} x$. Then $\exists$ a sequence of convex linear comb. of (x_k) which converges strongly to x, i.e.

$$\begin{cases} y_\ell = \sum_{k=1}^{N(\ell)} a_{k\ell} x_k \\ \sum_{n=1}^{N(\ell)} a_{n\ell} = 1 \end{cases} \quad \text{and} \quad y_\ell \xrightarrow{\ell \to \infty} x.$$

Pf. [recall if $K \subset X$ convex, closed then

$$K = \overline{K} = \overline{K}^w = \overline{K}^{w,seq}$$]

Apply ↑ to $K := \overline{\operatorname{conv}} \{ x_k : k \in \mathbb{N} \}$

by hyp. $x \in \overline{K}^{w,seq}$ so $x \in K$

then (by def of K) x is a limit point of convex linear combination, as claimed above. ∎

Existence of minima for ∞-dim'l problems

Must introduce two words:

$(X, \|\cdot\|)$ normed space $/\mathbb{R}$

$M \subset X$ subset $(M \neq \emptyset)$

$F: M \longrightarrow \mathbb{R}$

(1)

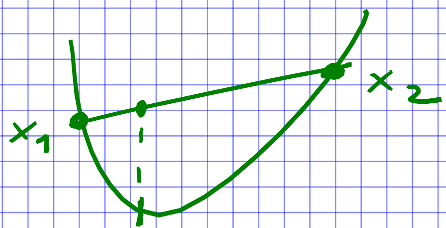
Def: F is weakly sequentially l.s.c. at $x_0 \in M$ if, for any sequence (x_k) in M w/ $x_k \xrightarrow{w} x_0$ one has
$$F(x_0) \leq \liminf_{k\to\infty} F(x_k).$$

Examples:

a) $M = X$, $F(x) = \|x\|$.

b) $M \subset X$ closed, convex, $F: M \longrightarrow \mathbb{R}$ continuous and convex

$$F\left(\sum_{i=1}^{i_0} \lambda_i x_i\right) \leq \sum_{i=1}^{i_0} \lambda_i F(x_i)$$



Why is any such F weakly sequentially l.s.c.?

Pf. $x_k \xrightarrow{w} x_0$, WLOG (possibly extracting a subsequence) ensure
$$F(x_k) \longrightarrow \alpha_0 := \liminf_{k\to\infty} F(x_k)$$
$(k \in \mathbb{N})$

Now, observe that for any $k_0 \in \mathbb{N}$, x_0 is (the) weak sequential limit of $(x_k)_{k \geq k_0}$

So $x_0 \in \overline{\text{conv}} \{ x_k : k \geq k_0 \}$

hence (by Mazur) $\exists \; y_\ell^{(k_0)} \longrightarrow x_0$

where $y_\ell^{(k_0)} = \sum_{k=k_0}^{N(\ell,k_0)} a_{k\ell}^{(k_0)} x_k$

w/ $\sum_{k=k_0}^{N(\ell,k_0)} a_{k\ell}^{(k_0)} = 1$ Hence

$$F\left(y_\ell^{(k_0)}\right) = F\left(\sum_{k=k_0}^{N(\ell,k_0)} a_{k\ell}^{(k_0)} x_k\right)$$

$$\overset{F \text{ convex}}{\leq} \sum_{k=k_0}^{N(\ell,k_0)} a_{k\ell}^{(k_0)} F(x_k)$$

$$\leq \sup_{k \geq k_0} F(x_k) \qquad (*)$$

Let $\ell \to \infty$, using continuity of F have

$$F\left(y_\ell^{(k_0)}\right) \longrightarrow F(x_0)$$

So letting $\ell \to \infty$ in $(*)$ get

$$F(x_0) \leq \sup_{k \geq k_0} F(x_k)$$

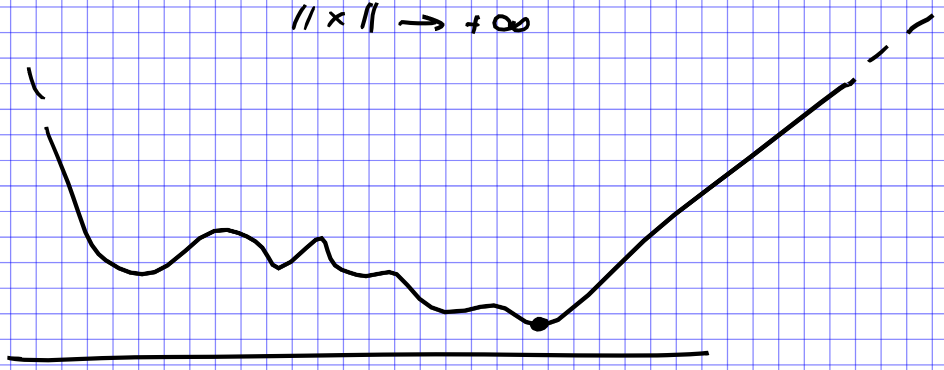
This inequality can be proven for any given $k_0 \in \mathbb{N}$:
if I let $k_0 \longrightarrow \infty$ and I recall one possible definition of limsup, we get

$$F(x_0) \leq \underbrace{\lim_{k_0 \to \infty} \left[\sup_{k \geq k_0} F(x_k)\right]}$$

$$\equiv \limsup_{k \to \infty} F(x_k) = \alpha_0$$

by extracting a subseq. at the beginning ∎

(2) **Def** We'll say that $F: M \to \mathbb{R}$ is coercive if
$$\lim_{\|x\| \to +\infty} F(x) = +\infty.$$



Thm. (direct methods) Let X be a reflexive normed space, $M \subset X$ ($M \neq \emptyset$) weakly sequentially closed, $F: M \longrightarrow \mathbb{R}$ coercive and weakly sequentially l.s.c.. Then $\exists\, x_0 \in M$ w/ $F(x_0) = \inf_{x \in M} F(x)$

(i.e. F has a global min on M).

Pf. Pick a minimising sequence (x_k), i.e.
$$F(x_k) \longrightarrow \inf_{x \in M} F(x)$$

F coercive $\Longrightarrow$ (x_k) bounded.

So Eberlein-Smulian $\exists$ (a subsequence...) $x_k \xrightarrow{w} x_0$

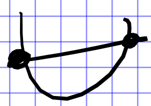
A priori $x_0 \in X$, but M weakly seq. closed $\Longrightarrow$ $x_0 \in M$.

Weak seq. l.s.c. $F(x_0) \le \liminf_{k \to \infty} F(x_k) = \inf_{x \in M} F(x)$

Conclusion: $\inf_{x \in M} F(x) \in \mathbb{R}$, and $x_0 \in M$ is a minimiser. ∎

Uniqueness : e.g. need M convex, F strictly convex

$x_0 \neq x_1$ w/ $F(x_0) = \inf_{x \in M} F(x) = F(x_1)$



look at $\dfrac{x_0+x_1}{2}$ and use convexity

$$F\left(\frac{x_0+x_1}{2}\right) < \frac{1}{2}\left(F(x_0)+F(x_1)\right) = \inf_{x\in M} F(x)$$

Rmk. (variational proof of Riesz)

Given $\ell \in H^*$, consider $F(x) := \frac{1}{2}\|x\|_H^2 - \ell(x)$

$F : H \longrightarrow \mathbb{R}$

Claim : F — coercive, continuous, convex

coercive: $F(x) \geq \frac{1}{2}\|x\|_H^2 - \|\ell\|_{H^*}\|x\|_H = \underbrace{\|x\|_H}_{\to +\infty}\underbrace{\left(\frac{\|x\|_H}{2} - \|\ell\|_{H^*}\right)}_{\to +\infty}$

continuous, convex $\Rightarrow$ weakly seq. l.s.c.

convex: triangle ineq.

$$\|t x_1 + (1-t)x_2\|_H^2 \leq \left(t\|x_1\| + (1-t)\|x_2\|\right)^2$$

$$\leq t\|x_1\|^2 + (1-t)\|x_2\|^2$$

convexity $s \mapsto s^2$

Can apply thm. above $\rightsquigarrow \exists\, y \in H$, a global min for F

(1st derivative test) i.e. compute Euler-Lagrange equation for F :

Given any $z \in H$ have $\epsilon \longmapsto F(y+\epsilon z)$

has a min at $\epsilon = 0$

thus $\left[\dfrac{d}{d\epsilon}\right]_{\epsilon=0} F(y+\epsilon z) = 0$

$= (y,z)_H - \ell(z)$

$\forall z \in H \implies \ell(z) = (y,z)_H$ ∎

$F(y+\epsilon z) = \frac{1}{2}\|y+\epsilon z\|^2 - \ell(y+\epsilon z)$

$= \frac{1}{2}\|y\|^2 + \frac{\epsilon^2}{2}\|z\|^2 + \epsilon\underbrace{(y,z)} - \ell(y) - \epsilon\underbrace{\ell(z)}$

Comments : similarly one can give a variational proof of

$$\begin{cases} \left(L^p(\Omega)\right)^* \cong L^q(\Omega) \\ 1 < p < \infty \end{cases}$$

Issue was :

$$J : L^q \longrightarrow (L^p)^*$$

$$g \longmapsto \int_\Omega g \,\cdot$$

surjective

To do that, given $\ell \in (L^p)^*$ consider $E : L^p \longrightarrow \mathbb{R}$ defined by $E(f) = \frac{1}{p} \|f\|_{L^p}^p - \ell(f)$

(check as above ...) $\rightsquigarrow \exists !$ minimizer $f_0 \in L^p$

Euler - Lagrange equation $0 = \int_\Omega |f_0|^{p-2} f_0 f - \ell(f)$

$$\forall f \in L^p(\Omega)$$

i.e. if we set $g := |f_0|^{p-2} f_0 \in L^q$

have $\ell(f) = \int_\Omega g f \quad \forall f \in L^p$

Achtung : both in this proof, and in the one above for Riesz we need to know that the space in question is REFLEXIVE

so these are NOT new / logically independent proofs of these theorems.