# Functional Analysis I – L22 30/11/2020

## The adjoint operator

$(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ normed spaces / $\mathbb{R}$

duals $(X^*, \|\cdot\|_{X^*})$, $(Y^*, \|\cdot\|_{Y^*})$ Banach

**Idea:**

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \Big| * & & \Big| * \\ X^* & \xleftarrow{T^*} & Y^* \end{array}$$

✓

if $T \in L(X,Y)$ then $T^* \in L(Y^*, X^*)$

defined by $\langle T^*(y^*), x\rangle = \langle y^*, Tx\rangle$

$\langle \, , \, \rangle_{X^* \times X}$ $\langle \, , \, \rangle_{Y^* \times Y}$

**Def**: given $A : D_A \subset X \longrightarrow Y$ linear w/ $\overline{D_A} = X$ (densely defined)

we set $A^* : D_{A^*} \subset Y^* \longrightarrow X^*$ for

$D_{A^*} = \{ y^* \in Y^* :$ $\begin{array}{rcl} D_A & \xrightarrow{\ell_{y^*}} & \mathbb{R} \\ x & \longmapsto & \langle y^*, Ax\rangle_{Y^* \times Y} \end{array}$ is continuous $\}$

and $A^* y^*$ is the (uniquely determined) extension of $\ell_{y^*}$ to $X$.

$$\frac{|\ell_{y^*}(x)|}{\|x\|_X} \overset{?}{\leq} C$$

**Key remark**

if $x \in D_A$, $y^* \in D_{A^*}$

$$\boxed{\langle A^* y^*, x\rangle_{X^* \times X} = \langle y^*, Ax\rangle_{Y^* \times Y}}$$

## Isometry Property

**Thm.** (setting above) $A \in L(X, Y) \Longrightarrow A^* \in L(Y^*, X^*)$

w/ same norm $\quad \|A\|_{L(X,Y)} = \|A^*\|_{L(Y^*,X^*)}$.

**Pf:** in this case (by def) $D_A = X$, and I claim

$\boxed{D_{A^*} = Y^*}$. why?

$$X \longrightarrow \mathbb{R}$$
$$x \longmapsto \langle y^*, Ax \rangle$$

$$|\langle y^*, Ax\rangle| \leq \|y^*\|_{Y^*} \cdot \|Ax\|_Y$$
$$\leq \|y^*\|_{Y^*} \|A\|_{L(X,Y)} \|x\|_X$$

$A \in L(X, Y)$

$\rightsquigarrow \quad l_{y^*} \in X^* \quad$ w/ $\quad \|l_{y^*}\|_{X^*} \leq \|y^*\|_{Y^*} \|A\|_{L(X,Y)}$

To check the isometry property: recall the key property reads

$$\begin{cases} \langle A^* y^*, x \rangle_{X^* \times X} = \langle y^*, Ax \rangle_{Y^* \times Y} \\ \forall x \in X, \ \forall y^* \in Y^* \end{cases}$$

By def

$$\|A^*\|_{L(Y^*, X^*)} = \sup_{\|y^*\|_{Y^*} \leq 1} \|A^* y^*\|_{X^*}$$

$$= \sup_{\|y^*\|_{Y^*} \leq 1} \Big( \sup_{\|x\|_X \leq 1} \underbrace{|\langle A^* y^*, x\rangle|}_{\text{now apply the key property}} \Big)$$

$$= \sup_{\|x\|_X \leq 1} \underbrace{\Big( \sup_{\|y^*\|_{Y^*} \leq 1} |\langle y^*, Ax\rangle| \Big)}_{\text{by dual char of norm } = \|Ax\|_Y}$$

$$= \sup_{\|x\|_X \leq 1} \|Ax\|_Y = \|A\|_{L(X,Y)} \quad \blacksquare$$

Prove generally:

**Prop.** $A : D_A \subset X \longrightarrow Y$ linear and densely defined i.e. $\overline{D_A} = X$

then $A^* : D_{A^*} \subset Y^* \longrightarrow X^*$ has closed graph.

**Pf.** pick a sequence of points on the graph of the adjoint

i.e. $$(y_k^*, A^* y_k^*) \in \Gamma_{A^*}$$

w/ $$y_k^* \longrightarrow y^* \in Y^*$$
$$A^* y_k^* := x_k^* \longrightarrow x^* \in X^*$$

want to check $\underbrace{y^* \in D_{A^*}}$ and $\underbrace{A^* y^* = x^*}$

**why true?** $$\langle y^*, Ax \rangle = \lim_{k\to\infty} \langle y_k^*, Ax \rangle$$
$$= \lim_{k\to\infty} \langle A^* y_k^*, x \rangle = \langle x^*, x \rangle$$

this implies that $|\langle y^*, Ax \rangle| \le \|x^*\|_{X^*} \underbrace{\|x\|_X}$

$\Longrightarrow$ $y^* \in D_{A^*}$ and $\langle y^*, Ax \rangle = \langle x^*, x \rangle$
$$\langle y^*, Ax \rangle = \langle A^* y^*, x \rangle$$

so $\langle (A^* y^* - x^*), x \rangle = 0$ true $\forall x \in D_A$ dense

$$\Longrightarrow A^* y^* = x^* \qquad \blacksquare$$

**Rmk.** Setting above, we have that

$$\underbrace{A \subset B} \Longrightarrow B^* \subset A^*$$

in the sense of inclusion of graphs i.e. $\Gamma_A \subset \Gamma_B$

For a number of extra properties of the adjoint see Problems 11.1 and 11.2

<u>why?</u> $A \subset B$ means $D_A \subset D_B$

w/ restriction property $B\big|_{D_A} = A$

let $y^* \in D_{B^*}$, pick $x \in D_A$

key prop.

$$\underbrace{\langle y^*, Ax \rangle_{Y^* \times Y}} = \langle y^*, Bx \rangle_{Y^* \times Y} = \underbrace{\langle B^* y^*, x \rangle_{X^* \times X}} = \langle A^* y^*, x \rangle$$

$\Longrightarrow$ $y^* \in D_{A^*}$ and

$A^* y^* = B^* y^*$ i.e. $A^*\big|_{D(B^*)} = B^*$ ∎

<u>Example:</u> $1 < p, q < \infty$ w/ $\frac{1}{p} + \frac{1}{q} = 1$, $\Omega \subset \mathbb{R}^n$ open

$X = Y = L^p(\Omega)$, <u>duals</u> $X^* = Y^* \cong L^q(\Omega)$

<u>Operator</u> $A : D_A \subset X \longrightarrow Y$

$D_A = C_c^\infty(\Omega)$

$A : f \longmapsto \Delta f$

well-defined, <u>linear</u>

What is the adjoint?

$D_{A^*} = \{ g \in Y^* \cong L^q(\Omega) :$ $\underset{\in D_A \subset L^p}{f} \longmapsto \langle g, \Delta f \rangle = \int_\Omega g(x) \Delta f(x)$ is continuous $\}$

if $f, g \in C_c^\infty(\Omega)$

but $$\int_\Omega g(x) \Delta f(x)\, dx \underset{\text{by parts}}{=} \int_\Omega \Delta g(x) f(x) dx \qquad (\$)$$

Using $(\$)$ have that

$$|\langle g, \Delta f \rangle| \underset{\text{Hölder}}{\leq} \|\Delta g\|_{L^q} \|f\|_{L^p}$$

thus the map $f \longmapsto \langle g, \Delta f \rangle$ is extendable to a bounded linear functional on all of $L^p$ hence $D_{A^*} \supset C_c^\infty(\Omega)$

and $\forall g \in C_c^\infty(\Omega)$ $A^* g = \Delta g$

we'll say: $\Delta$ is a self-adjoint operator

# Utility of $T^*$: solvability of linear equations

Two points:

① the goal now is to solve an equation of the form
$$Ax = y$$
in an infinite-dimensional context;

② "homogeneous equations are much simpler than inhomogeneous ones"

**Prop.** Let $X, Y$ be Banach spaces, $A : D_A \subset X \longrightarrow Y$ linear, densely defined and closed. TFAE:
($\overline{D_A} = X$) ($\Gamma_A \subset X \times Y$ closed)

i) $\mathrm{im}\,(A)$ closed in $Y$ ✓

ii) $\mathrm{im}\,(A^*)$ closed in $X^*$

iii) $\mathrm{im}\,(A) = {}^{\perp}\ker(A^*) = \{ y \in Y : \langle y^*, y \rangle = 0 \ \forall\, y^* \in \ker(A^*) \}$

iv) $\mathrm{im}\,(A^*) = \ker(A)^{\perp} = \{ x^* \in X^* : \langle x^*, x \rangle = 0 \ \forall\, x \in \ker(A) \}$

**Rmk. 1** in practice, one (most often) works w/ bounded operators (i.e. $A \in L(X, Y)$) for which property i) is true [examples: Fredholm operator, see Thursday's lecture]

Thus iii) and iv) are TRUE as well. In part.
$$\mathrm{im}\,(A) = {}^{\perp}\ker(A^*)$$
i.e. $Ax = y$ is solvable iff $y \in {}^{\perp}\ker(A^*)$.

**Thm R. 2** For brevity, we'll only prove i) $\Longleftrightarrow$ iii).

**Lemma** $(X, \|\cdot\|_X)$ normed space. Then

(a) $M \subset X$ subspace, then $M^\perp \subset X^*$ **closed**

(actually $M^\perp$ is $w^*$- seq. closed)

(b) $L \subset X^*$ subspace, then ${}^\perp L \subset X$ **closed**

(actually ${}^\perp L$ is $w$ - seq. closed)

**Pf:** let's do (b), note (a) is identical.

$(x_k) \subset {}^\perp L$ w/ $x_k \xrightarrow{w} x$ $(k \to \infty)$

means: $\forall\, x^* \in X^*$ $\langle x^*, x\rangle = \lim_{k\to\infty} \langle x^*, x_k\rangle$

but by hp. $\langle x^*, x_k\rangle = 0$ $(\forall k \in \mathbb{N})$

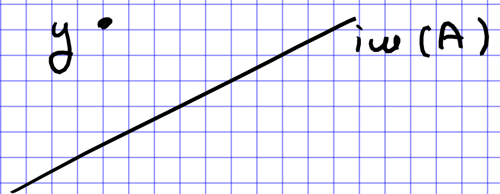
$\Longrightarrow$ $\langle x^*, x\rangle = 0$ $\Longrightarrow$ $x \in {}^\perp L$. ■

**Pf:** iii) $\Longrightarrow$ i) follows straight from the lemma

i) $\Longrightarrow$ iii) recall that $\forall x \in D_A$, $\forall y^* \in D_{A^*}$

$$\langle A^* y^*, x\rangle = \langle y^*, Ax\rangle$$

thus $\mathrm{im}(A) \subset {}^\perp\ker(A^*)$. Equality gained arguing by contradiction: let $y \in {}^\perp\ker(A^*) \setminus \mathrm{im}(A)$




$\Longrightarrow$ $\exists\, y^* \in Y^*$ w/

$y^*|_{\mathrm{im}(A)} = 0$

$\boxed{\langle y^*, y\rangle \neq 0}$

$y^*|_{\mathrm{im}(A)} = 0$ means $\boxed{\langle y^*, Ax \rangle = 0 \quad \forall x \in D_A}$

$\Downarrow$

$y^* \in D_{A^*}$ w/ $A^* y^* = 0$

$\Longrightarrow$ $y^* \in \ker(A^*)$

But $y \in {}^{\perp}\ker(A^*) \Longrightarrow \langle y^*, y \rangle = 0$

CONTRADICTION ∎

<u>A criterion for the solvability of equations:</u>

<u>Prop.</u> (setup as above) $A : D_A \subset X \longrightarrow Y$. TFAE:

i) $A$ surjective

ii) $A^*$ injective and $\mathrm{im}(A^*)$ closed ← see Thursday's lecture

iii) $\exists\, c_0 > 0 \quad \forall y^* \in D_{A^*} : \ c_0 \|y^*\|_{Y^*} \le \|A^* y^*\|_{X^*}$

(i.e. $A^*$ is uniformly coercive)

<u>Pf.</u> i) $\Longleftrightarrow$ ii) follows straight from previous thm.

ii) $\Rightarrow$ iii) $A^* : D_{A^*} \subset Y^* \longrightarrow \underbrace{\mathrm{im}(A^*)}_{\text{Banach space}}$

in this situation I can invoke <u>Satz 3.3.2</u>

(Satz von Stetigen Inversen)

$\Longrightarrow \exists\, (A^*)^{-1} \in L(\mathrm{im}(A^*); Y^*)$

in part. this means that $\boxed{\forall x^* \in \mathrm{im}(A^*)}$

$\|(A^*)^{-1} x^*\| \le C \|x^*\|$

equivalently $\forall y^* \in D_{A^*} \subset Y^*$

$\boxed{x^* = A^* y^*}$ and so, if I plug in the inequality above I get

$$\| y^* \|_{Y^*} \leq C \, \| A^* y^* \|_{X^*} .$$

iii) $\Rightarrow$ ii) $A^*$ injective is obvious!

Let's check im $(A^*)$ is closed.

$x_k^* = A^* y_k^*$ $\qquad x_k^* \xrightarrow[(k \to \infty)]{} x^*$ $\stackrel{(iii)}{\Longrightarrow}$ $(y_k^*)$ Cauchy in $Y^*$

so $\qquad y_k^* \longrightarrow y^*$. By closed graph property
($Y^*$ Banach) (of $A^*$)

$$(y_k^*, A^* y_k^*) \in \Gamma_{A^*} \Longrightarrow (y^*, x^*) \in \Gamma_{A^*}$$

i.e. $y^* \in D_{A^*}$, $x^* = A^* y^*$ ∎