Functional Analysis I - L23 3/12/2020

Compact Operators

<u>Setup</u>: $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ $L(X,Y)$

<u>Def</u>: $T \in L(X,Y)$ is called <u>compact</u>
if $\overline{T(B_1(0;X))}$ is compact in $(Y, \|\cdot\|_Y)$.

this is a "rare" property for an operator

<u>Rmk.</u> if $\dim X < \infty$ this notion is insignificant ...

Some basic facts about compact operators
(see also <u>problem 11.4</u> for additional ones):

<u>Lemma</u> Let $T \in L(X,Y)$ be <u>compact</u>, then
$$x_k \xrightarrow{w} x \implies Tx_k \longrightarrow Tx$$

<u>Pf.</u> • $T \in L(X,Y) \rightsquigarrow$ $x_k \xrightarrow{w} x \Rightarrow Tx_k \xrightarrow{w} Tx$
(cf. Problem 9.3)

• $x_k \xrightarrow{w} x \implies (x_k)$ bounded in X
i.e. $\exists R > 0$ s.t. $(x_k) \subset B_R(0;X)$

$\underset{\text{(compactness hp.)}}{\implies}$ (Tx_k) has a converging subsequence
i.e. $\underset{k \in \Lambda \subset \mathbb{N}}{Tx_k} \longrightarrow y \in Y$

in part. $\underset{k \in \Lambda}{Tx_k} \xrightarrow{w} y$

by uniqueness of the weak limit $\implies y = Tx$
(so far we have: a <u>subsequence</u> of (Tx_k) converges strongly to Tx)

- but actually: the argument above shows that any subsequence of (x_k) has a converging subsequence such that the images through T converge strongly to Tx, in other words: $Tx \in Y$ is the ONLY LIMIT POINT of the sequence (Tx_k) in Y.

  This means $Tx_k \longrightarrow Tx$ ∎

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Thm. Let X be Banach and let $T \in \mathcal{L}(X)$ be compact. Then the image of $\mathrm{id} - T : X \circlearrowleft$ is closed.

(thus we can apply "orthogonality relations" proven in Lecture 22)

Pf. (intermediate claims are results of independent relevance!)

Claim 1: $\ker(\mathrm{id} - T) =: M$ is finite dimensional.

why?

$$\mathrm{id}|_M = T|_M \Longrightarrow \overline{B_1(0;M)} = \overline{TB_1(0;M)} \underset{\text{since } M \subset X}{\subset} \underbrace{\overline{TB_1(0;X)}}_{\text{compact by hp.}}$$

Thus: M is a Banach space which has relatively compact unit ball

$\Longrightarrow \dim M < \infty$

Claim 2: M has a $\underbrace{\text{topological complement}}_{\text{cf pb. 3.4 and 7.2}}$

i.e. $X = M \oplus L$ for some closed subspace L.

Claim 3: set $S := \mathrm{id} - T : L \longrightarrow X$

satisfies a coercivity estimate, i.e. $\exists\, r > 0$ s.t.

$\forall x \in L \quad \boxed{r \|x\|_X \leq \|Sx\|_X}$

why? if not $\exists\ (x_k)$ in $L \subset X$ w/ $\|x_k\|_X = 1$
$\|Sx_k\|_X \le 1/k$
$(<)$

$$\begin{array}{ccccc} Sx_k & = & x_k & - & Tx_k \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & x_0 & & x_0 \in L \end{array}$$

$\rightsquigarrow$ for some subsequence $\Lambda \subset \mathbb{N}$

$x_k \longrightarrow x_0 \overset{T \in L(X,Y)}{\Longrightarrow} Tx_k \longrightarrow Tx_0$, $\quad Tx_k \longrightarrow x_0$ $\quad \Longrightarrow \boxed{x_0 = Tx_0}$

$\Longrightarrow\ x_0 \in \mathrm{Ker}(S) = M \rightsquigarrow x_0 \in L \cap M \Longrightarrow x_0 = 0$
$X = M \oplus L$

this fact contradicts $\|x_k\|_X = 1$ and $x_k \underset{k \in \Lambda}{\longrightarrow} x_0$

<u>Claim 4</u>: $\mathrm{im}(S) = \mathrm{im}(id - T)$ <u>is closed</u> in X.

why? $y_k = Sx_k \longrightarrow y \Longrightarrow (y_k)$ Cauchy
$\Downarrow$ $\boxed{\checkmark\checkmark\checkmark}$
$\underline{?}.\ y \overset{?}{\in} \mathrm{im}(S)$ $\quad (x_k)$ Cauchy
$\Downarrow$
(x_k) converges
$x_k \longrightarrow x$
$Sx_k \longrightarrow Sx$

uniqueness of the limit $\rightsquigarrow y = Sx \rightsquigarrow y \in \mathrm{im}(S)$ ■

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<u>Next</u>: <u>examples</u> of compact operators

Compactness criteria:

(A) (Arzelà - Ascoli)

$\Omega \subset \mathbb{R}^n$ open w/ $\bar{\Omega}$ compact

$\mathcal{F} \subset C^0(\bar{\Omega})$ subclass. TFAE:

i) $\mathcal{F}$ is (relatively) sequentially compact, i.e.

given (f_ℓ) in $\mathcal{F}$ $\exists$ subsequence $(f_{\ell(n)})$

and $f \in C^0(\bar{\Omega})$ w/ $f_{\ell(n)} \xrightarrow{C^0(\bar{\Omega})} f$

ii) $\mathcal{F}$ is uniformly bounded i.e. $\sup_{f \in \mathcal{F}} \|f\|_{C^0} < \infty$

uniformly continuous i.e.

$\forall \epsilon > 0 \; \exists \delta > 0 \quad \forall x, y \in \Omega \qquad |x-y| < \delta$

w/ $\quad \forall f \in \mathcal{F} \qquad \Downarrow$

$|f(x) - f(y)| < \epsilon$

Example (A)

$X = C^1([0,1]) \qquad \|\cdot\|_X = \|\cdot\|_{C^1}$

$Y = C^0([0,1]) \qquad \|\cdot\|_Y = \|\cdot\|_{C^0}$

$T: X \longrightarrow Y$ the "natural embedding" i.e. $Tf = f$

is COMPACT.

Proof: i) $\Rightarrow$ ii) exercise: argue by contradiction!

ii) $\Longrightarrow$ i) pick (f_n) seq. in $\mathcal{F}$ $\leftarrow$ assumed uniformly bounded / continuous

GOAL $\to$ extract a converging subsequence in $C^0(\bar{\Omega})$

Strategy: similar to proof of Banach-Alaoglu....

Fix a dense set of points in $\Omega \subset \mathbb{R}^4$, $(x_\ell)_{\ell \in \mathbb{N}}$.

Can ensure convergence of (f_k) at grid points, i.e.

$$\exists\ \Lambda_1 \subset \mathbb{N} \text{ s.t. } \quad f_k(x_1) \longrightarrow a_1 \quad \begin{pmatrix} k \to \infty \\ k \in \Lambda_1 \end{pmatrix}$$

$$\cup$$

$$\exists\ \Lambda_2 \subset \mathbb{N} \text{ s.t. } \quad f_k(x_2) \longrightarrow a_2 \quad \begin{pmatrix} k \to \infty \\ k \in \Lambda_2 \end{pmatrix}$$

get $\Lambda_1 \supset \Lambda_2 \supset \Lambda_3 \supset \ldots$ Then "extract a diagonal sequence"

get $\Lambda \subset \mathbb{N}$ unbounded s.t.

$$\forall\ \ell \in \mathbb{N} \qquad f_k(x_\ell) \longrightarrow a_\ell \qquad \begin{pmatrix} k \to \infty \\ k \in \Lambda \end{pmatrix}$$

Define the "candidate limit function"

$$f(x_\ell) = a_\ell \qquad (\forall\ \ell \in \mathbb{N})$$

<u>Claim 1</u>: $f : D \longrightarrow \mathbb{R}$, $D = \bigcup_{\ell \in \mathbb{N}} \{x_\ell\}$ grid points is <u>uniformly continuous</u> (hence $\exists!$ extension $f \in C^0(\overline{\Omega})$)

why? (3ϵ-trick) given $\epsilon > 0$ let $\delta = \delta(\epsilon) > 0$ s.t.

$$|x - y| < \delta \implies |f_k(x) - f_k(y)| < \epsilon \quad (\forall k \in \mathbb{N})$$

Then pick $x_\ell, x_m \in D$ w/ $|x_\ell - x_m| < \delta$ so

$$|f(x_\ell) - f(x_m)| \leq |f(x_\ell) - f_k(x_\ell)| + \underbrace{|f_k(x_\ell) - f_k(x_m)|} + |f_k(x_m) - f(x_m)|$$

$$\leq \epsilon + \epsilon + \epsilon = 3\epsilon$$

$k \geq k_0$

<u>Claim 2</u>: $f_k \longrightarrow f$ in $C^0(\overline{\Omega})$ $\begin{pmatrix} k \to \infty \\ k \in \Lambda \end{pmatrix}$

why? use $\overline{\Omega}$ compact: <u>finite</u> covering of $\overline{\Omega}$ of the form $B_\delta(x_\ell)$ x_ℓ grid point, $1 \leq \ell \leq L$

Now choose k_* s.t. $k \geq k_*$ ← is independent of $x \in \Omega$ (only determined by the cover)

$$|f_k(x_\ell) - f(x_\ell)| < \epsilon \qquad \forall\ 1 \le \ell \le L$$

Given $x \in \Omega$ pick $1 \le \ell \le L$ w/ $x \in B_\delta(x_\ell)$

$$\begin{aligned} |f_k(x) - f(x)| &\le |f_k(x) - f_k(x_\ell)| \\ &\quad + |f_k(x_\ell) - f(x_\ell)| \\ &\quad + |f(x_\ell) - f(x)| \\ &\le \epsilon + \epsilon + \epsilon = 3\epsilon \end{aligned}$$ ■

③ A compactness theorem in L^p (Fréchet-Kolmogorov)

$\Omega \subset \mathbb{R}^n$ open, bounded. $1 \le p < \infty$

$\mathcal{F} \subset L^p(\Omega)$ subclass. TFAE:

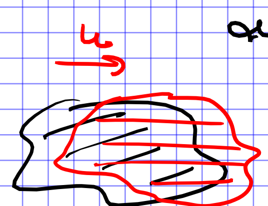
i) $\mathcal{F}$ is (relatively) sequentially compactness;

ii) $\mathcal{F}$ is uniformly bounded in L^p, i.e. $\sup_{f \in \mathcal{F}} \|f\|_{L^p} < \infty$

and uniformly continuous in L^p, i.e.

$$\sup_{f \in \mathcal{F}} \|f - \tau_h f\|_{L^p(\mathbb{R}^n)} \xrightarrow{h \to 0} 0$$

notation: given $f \in \mathcal{F}$ extend it to 0 outside of Ω and $\tau_h f(x) = f(x+h)$



Pf. i) $\Rightarrow$ ii) $\bar{\mathcal{F}}$ seq. compact $\overset{\text{Lecture 5}}{\Longleftrightarrow}$ $\bar{\mathcal{F}}$ compact

$\Longrightarrow$ $\bar{\mathcal{F}}$ bounded

Must check the other property: we use that $C_c^\infty(\mathbb{R}^4)$ is dense in $L^p(\mathbb{R}^4)$ if $p<\infty$, so given any $\epsilon>0$

$$\bigcup_{f\in C_c^\infty(\mathbb{R}^4)} B_\epsilon\left(f|_\Omega ; L^p(\Omega)\right) = L^p(\Omega)$$

hence $\exists$ finite covering w/ pivots $f_1, f_2, \dots, f_K$

so $\exists\, \delta>0$ w/ $|x-y|<\delta \Rightarrow |f_i(x)-f_i(y)|^p < \dfrac{\epsilon^p}{\mathcal{L}^4(\mathrm{supp}(f_i))}$

$\forall\, 1\le i\le K$

It follows that

$h\in\mathbb{R}^4$, $|h|<\delta$ $\quad \|f_i-\tau_h f_i\|^p_{L^p(\mathbb{R}^4)} \le \|f_i-\tau_h f_i\|^p_{L^\infty}\displaystyle\int_{\mathrm{supp}(f_i)} 1 < \epsilon^p$

$\rightsquigarrow \quad \|f_i-\tau_h f_i\|_{L^p} < \epsilon$

$i\in\{1,\dots,k\}$

Given $f\in\mathcal{F}$ have $f\in B_\epsilon(f_k)$ and at this stage we prove "uniform continuity in L^p" using the 3ϵ-trick as above.

ii) $\Rightarrow$ i) (done by smooth approximation)

$\rho\in C_c^\infty(B_1(0))$ $\qquad \rho_\delta(x) := \delta^{-4}\rho\left(\frac{x}{\delta}\right)$ $\quad (\delta>0)$

$\int\rho = 1$ $\qquad \left[\int\rho_\delta = 1\right]$

$$f_\delta := f*\rho_\delta = \int_{\mathbb{R}^4} f(x-y)\,\rho_\delta(y)\,dy$$

Given $\epsilon>0$ choose (and fix) $\delta>0$ s.t.

$\|f-\tau_h f\|_{L^p}<\epsilon$

$\forall f\in\mathcal{F}$ (provided $|h|<\delta$).

Claim 1: $\quad \|f-f_\delta\|_{L^p}<\epsilon$

why? $(f_\delta - f)(x) = \int_{\mathbb{R}^n} (f(x-y) - f(x))\rho_\delta(y)\,dy$

$\rightsquigarrow \|f_\delta - f\|_{L^p} \leq \int_{\mathbb{R}^n} \underbrace{\|\tau_y f - f\|_{L^p}}\rho_\delta(y)\,dy$

Minkowski

$\operatorname{supp}(\rho_\delta) \subset B_\delta(0;\mathbb{R}^n)$

$y \in B_\delta(0;\mathbb{R}^n)$

$\leq \int_{\mathbb{R}^n} \epsilon \cdot \rho_\delta(y)\,dy = \epsilon$ $\blacksquare$

<u>Claim 2</u>: $(f_\delta)_{f\in\mathcal{F}} \subset C^0(\bar\Omega)$ <u>uniformly</u> $\begin{cases}\text{bounded}\\ \text{cont.}\end{cases}$

why? $|f_\delta(x)| \leq \int_{\mathbb{R}^n} |f(x-y)|\,|\rho_\delta(y)|\,dy$

$L^1-L^\infty \to \leq C_\delta \int_{B_\delta} |f(x-y)|\,dy$

Hölder $\to \leq C'_\delta \int_{B_\delta} |f(x-y)|^p\,dy$

$\leq C'_\delta \|f\|^p_{L^p} \leq$ constant dep. on δ

$|f_\delta(x) - f_\delta(z)| = \left|\int_{\mathbb{R}^n} \underbrace{(\rho_\delta(x-y) - \rho_\delta(z-y))}_{\leq |\nabla\rho_\delta|\,|x-z|} f(y)\,dy\right|$

$\leq C_\delta\,|x-z|\,\|f\|^p_{L^p} \leq$ constant dep. on δ

<u>Close</u> $(f_k) \rightsquigarrow (f_{k,\delta})$

(given $\epsilon>0$ arbitrary and $\delta>0$ chosen as per Claim 1)

$\exists\ \Lambda = \Lambda(\delta) \subset \mathbb{N}$ unbounded s.t. {by Ascoli-Arzelà}

$f_{k,\delta} \xrightarrow{C^0(\bar\Omega)} f \quad \left(\begin{matrix}k\to\infty\\ k\in\Lambda\end{matrix}\right)$

now $\exists\ k_0 \in \mathbb{N}$ s.t. $j,k \geq k_0$ and $j,k\in\Lambda$

$$\| f_{j,\delta} - f_{k,\delta} \|_{L^p} < \epsilon$$

<u>thus</u> $\| f_j - f_k \|_{L^p} \leq \| f_j - f_{j,\delta} \|_{L^p} + \| f_{j,\delta} - f_{k,\delta} \|_{L^p} + \| f_{k,\delta} - f_k \|_{L^p}$

again $j, k \geq k_0$, $j, k \in A$

$$\leq \epsilon + \epsilon + \epsilon = 3\epsilon$$

$\rightsquigarrow$ I've shown that $(f_k) \subset L^p(\Omega)$ has a Cauchy subsequence $\underset{\text{since } L^p \text{ is Banach}}{\rightsquigarrow}$ the same subsequence converges

$\blacksquare$

Example B

Integral Kernels

Motivation/context: consider the Dirichlet problem

$$(\#)\begin{cases} \Delta u = f & \Omega \\ u = 0 & \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ open, bounded w/ smooth boundary $\partial\Omega$.

Given $f \in C^0(\overline{\Omega})$ are there solutions $u \in C^2(\overline{\Omega})$ to $(\#)$? <u>Unique</u> solution or not? The answer is <u>YES</u> (to both) and one can <u>represent</u> the (only) sol. as

$$u(x) = \int_\Omega \underbrace{G(x,y)}_{\text{Green function for } \Delta \text{ w/ Dirichlet boundary conditions.}} f(y)\, dy$$

The Green function $G(x,y)$ is an example of <u>integral kernel</u>.

It is possible e.g. to prove that the linear map

$$T: L^2(\Omega) \longrightarrow L^2(\Omega)$$

given by

$$Tf(x) = \int_\Omega G(x,y) f(y)\, dy$$

is <u>well-defined</u> and <u>compact</u> (cf. exercise 11.5 for a special case, and FA 2).