Functional Analysis I - L 24 7/12/2020

$\boxed{\text{Adjunction in Hilbert spaces}}$

Recall:

$$\begin{array}{ccc} X & \xrightarrow{A} & Y \\ \big\downarrow * & & \big\downarrow * \\ X^* & \xleftarrow{A^*} & Y^* \end{array}$$

Key identity: ✓

$$\langle y^*, Ax\rangle_{Y^*\times Y} = \langle A^* y^*, x\rangle_{X^*\times X}$$

$\forall x \in D_A$, $\forall y^* \in D_{A^*}$

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Hilbert space setting $\rightsquigarrow$ the story simplifies a lot

$(H, \underbrace{(\cdot,\cdot)})$ over $\mathbb{R}$. $\quad J : H \longrightarrow H^*$ ← (surjective) isometry

Given $A : D_A \subset H \longrightarrow H$

$[\ A^* : D_{A^*} \subset H^* \longrightarrow H^*$, cf. Lecture 22 $]$

Def (adjoint) $A^T : D_{A^T} \subset H \longrightarrow H$

$D_{A^T} = \{ y \in H$ s.t. $\quad D_A \xrightarrow{\ell_y} \mathbb{R}$

$\qquad\qquad\qquad\qquad x \longmapsto (y, Ax)$

is continuous $\}$ i.e. $\displaystyle\sup_{\substack{x\in D_A \\ \|x\| \le 1}} |(y, Ax)| < C$

If $y \in D_{A^T}$, denoted by $\ell_y \in H^*$ the (unique) extension of the continuous map above we set $A^T y := J^{-1} \ell_y$.

Key identity: $\quad (y, Ax) = (A^T y, x)$

$\forall x \in D_A \quad \forall y \in D_{A^T}$

Link:

i) $y^* \in D_{A^*} \iff J^{-1} y^* \in D_{A^T}$

ii) $\forall\, y^* \in D_{A^*}$ $\quad A^* y^* = J A^T J^{-1} y^*$

i.e. $\boxed{A^* = J \circ A^T \circ J^{-1}}$

Example (verify check)

$H = \mathbb{R}^n$, standard Euclidean product $\langle\,,\rangle$

have that $D_A = D_{A^T} = \mathbb{R}^n$ for any linear map $A: H \circlearrowleft$ (since linear $\Rightarrow$ continuous for finite-dim vector spaces)

why?

and A^T is represented by the transpose of the matrix representing A.

key identity $\boxed{\forall\, x, y \in \mathbb{R}^n}$ $\langle y, Ax\rangle = \langle A^T y, x\rangle$

$$\langle y, Ax\rangle = \sum_{i,j=1}^{n} (a_{ij} x_j)\, y_i = \sum_{i,j=1}^{n} (a_{ij} y_i)\, x_j = \boxed{\sum_{i,j=1}^{n} (a_{ji} y_j)\, x_i}$$

$$\langle A^T y, x\rangle = \boxed{\sum_{i,j=1}^{n} (a^T_{ij} y_j)\, x_i}$$

rename indices $(i,j) \longrightarrow (j,i)$

$\Longrightarrow a_{ji} = a^T_{ij}$ $\forall\, i,j$

Def (setting above) [From now onwards, we'll write A^* for A^T meaning the adjoint operator, $A^*: H \to H$

i) $A: D_A \subset H \longrightarrow H$ is called symmetric

if $A \subset A^*$ i.e. $D_A \subset D_{A^*}$ and $A^*|_{D_A} = A$

in particular $\langle y, Ax\rangle = \langle Ay, x\rangle$ $\forall\, x, y \in D_A$

ii) $A : D_A \subset H \longrightarrow H$ is called self-adjoint

if $A = A^*$ i.e. A sym and $D_A = D_{A^*}$.

Rmk. $\boxed{A \text{ sym}}$ $\not\Longrightarrow$ $\boxed{A \text{ self-adjoint}}$; $\boxed{A \text{ self-adjoint}} \Longrightarrow \boxed{A \text{ sym}}$

Rmk. 2 if $A \in L(H)$ then $A^* \in L(H)$

(cf. lecture 22), and so

$$A \text{ sym} \iff A \text{ self-adjoint}$$

Rmk. 3 Special case of Rmk. 2: if $T \in L(H)$ is self-adjoint, then $(id - T)$ is also self-adjoint.

If $T \in L(H)$ is self-adjoint and compact then the operator $S := id - T \in L(H)$ satisfies the following 3 properties:

1. S has closed image
2. S has finite-dim kernel
3. S has finite-dim cokernel

any bounded linear operator satisfying these properties is called Fredholm.

use orthogonality relations (fill in the gaps!)

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Example: a symmetric operator that is not self-adjoint:

$$D_{A_\infty} := C_c^\infty(\Omega) \subset \underbrace{L^2(\Omega)}_{H} \xrightarrow{A_\infty = \Delta} \underbrace{L^2(\Omega)}_{H}$$

$D_{A_1} := C^2(\overline{\Omega})$

$D_{A_2} := \{ u \in C^2(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega \}$

$D_{A_3} := \{ u \in C^2(\overline{\Omega}) : u = \partial u / \partial n = 0 \text{ on } \partial\Omega \}$

$$A_\infty \subsetneq A_3 \subsetneq A_2 \subsetneq A_1$$

$$A_1^* \subseteq A_2^* \subseteq A_3^* \subseteq A_\infty^*$$

Claim: A_3 is symmetric: it's enough to check that for all $u, v \in D_{A_3}$ $(u, A_3 v)_{L^2} = (A_3 u, v)_{L^2}$

i.e.

$$\boxed{\int_\Omega u(x)\, \Delta v(x)\, dx = \int_\Omega \Delta u(x)\, v(x)\, dx}$$

this is true! (apply div theorem and note that the boundary terms vanish by the way we defined D_{A_3})

But in fact $A_3 \subsetneq A_3^*$ because $\boxed{//}$ holds also if $u \in D_{A_3}$ and $v \in D_{A_1} \equiv C^2(\bar\Omega)$

so $\underbrace{D_{A_3^*} \supseteq D_{A_1} \supsetneq D_{A_3}}_{\supsetneq}$ $\rightsquigarrow$ A_3 is NOT self-adjoint.

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Spectrum and Resolvent

$(X, \|\cdot\|)$ Banach space $/ \mathbb{C}$, $A : D_A \subset X \longrightarrow X$ linear.

Def: The resolvent set of A is

$$\rho(A) = \{ \lambda \in \mathbb{C} : (\lambda - A) : D_A \longrightarrow X \text{ is bijective w/ bounded inverse } (\lambda - A)^{-1} \in L(X) \}$$

The spectrum of A is $\sigma(A) := \mathbb{C} \setminus \rho(A)$.

Example: $\dim_{\mathbb{C}} X < \infty$ $\rightsquigarrow$ $\sigma(A) = \{\text{eigenvalues of } A\}$ ← a bunch of points

Rmk. 1: if $A \in \mathcal{L}(X)$ then, using open mapping thm., $\rho(A) := \{\lambda \in \mathbb{C} : (\lambda - A) : X \circlearrowleft \text{ bijective}\}$.

Rmk. 2: $\left.\begin{array}{l}\text{if } A \text{ has closed graph} \\ \text{and } (\lambda - A) \text{ bijective}\end{array}\right\} \xRightarrow[\text{(Prop. 3.3.2)}]{} \exists\, (\lambda - A)^{-1} \in \mathcal{L}(X)$

So can simply def. of $\rho(A)$, as in Rmk. 1, under hp. A has closed graph.

Rmk. 3 (cf. Rmk. 2)

if $\rho(A) \neq \emptyset$ then $A : D_A \longrightarrow X$ has <u>closed graph</u>.

<u>(why?)</u> let $\lambda \in \rho(A)$, so $(\lambda - A)^{-1} \in \mathcal{L}(X)$

then $\Gamma_{(\lambda - A)^{-1}}$ is <u>closed</u>, i.e.

$$M := \{(x, y) \in X \times X : x = (\lambda - A)^{-1} y,\ y \in X\}$$

is closed in $X \times X$. Equivalently

$$M = \{(x, y) \in X \times X : x \in D_A \ \ y = (\lambda - A)x\}$$

thus $(\lambda - A)$ is <u>closed</u> (i.e. a <u>closed operator</u>)

But

$$A = \underbrace{(A - \lambda)}_{\text{closed}} + \underbrace{\lambda}_{\text{closed}}$$

<u>closed!</u>

Def: In the setting above, the <u>resolvent</u> of A is the map

$$\begin{aligned} R : \rho(A) &\longrightarrow \mathcal{L}(H) \\ \lambda &\longmapsto (\lambda - A)^{-1}. \end{aligned}$$

Thm. $A : D_A \subset X \longrightarrow X$ linear, $z_0 \in \rho(A)$

Then $\rho(A) \supset B_{\mathbb{C}}\left(z_0 ; \|R_{z_0}\|^{-1}_{L(X)}\right)$

thus $\rho(A) \subset \mathbb{C}$ is <u>open</u> ($\Leftrightarrow$ $\sigma(A)$ is <u>closed</u>)

and $\forall z \in \rho(A)$ $\quad \operatorname{dist}(z, \sigma(A)) \geq \dfrac{1}{\|R_z\|_{L(X)}}$.

Lastly, $\rho(A) \longrightarrow L(X)$, $z \longmapsto R_z$ is <u>continuous</u>.

Pf.

$$\left\{\begin{aligned} z - A &= (z - z_0) + (z_0 - A) \\ &= \underbrace{(1 + (z - z_0) R_{z_0})}\,(z_0 - A) \end{aligned}\right.$$

if $z_0 \in \rho(A)$ and $z \in B_{\mathbb{C}}\left(z_0, \|R_{z_0}\|^{-1}_{L(X)}\right)$

then (Neumann series) $\exists \; (1 + (z - z_0) R_{z_0})^{-1} \in L(X)$

and so $\exists \; R_z \in L(X)$ i.e. $z \in \rho(A)$ and

by $\{$
$$\begin{aligned} R_z &= (z_0 - A)^{-1} (1 + (z - z_0) R_{z_0})^{-1} \\ &= R_{z_0} (1 + (z - z_0) R_{z_0})^{-1} \end{aligned}$$

We only need to check that the resolvent (map) is continuous:

$$\begin{aligned} \|R_z - R_{z_0}\|_{L(X)} &\underset{\substack{\uparrow \\ \text{above}}}{=} \left\| R_{z_0} \left((1 + (z - z_0) R_{z_0})^{-1} - 1 \right) \right\|_{L(X)} \\ &\leq \|R_{z_0}\|_{L(X)} \, \| \underbrace{(1 + (z - z_0) R_{z_0})^{-1}}_{\xrightarrow{z \to z_0} \; 1} - 1 \|_{L(X)} \end{aligned}$$

use explicit form of the inverse as Neumann series

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Basic Resolvent Identities:

given $\lambda, \mu \in \rho(A)$ then

i) $R_\lambda A \subset A R_\lambda = \lambda R_\lambda - id_X \in L(X)$

ii) $R_\lambda - R_\mu = (\mu - \lambda) R_\mu R_\lambda$

iii) $R_\lambda R_\mu = R_\mu R_\lambda$.

Pf: i) "we're working in a commutative subalgebra, but we need to be careful with the domains"

$$\underbrace{R_\lambda (\lambda - A)} = R_\lambda \lambda - R_\lambda A = \lambda R_\lambda - R_\lambda A$$

$$= id_{D_A} \subset id_X = \underbrace{(\lambda - A) R_\lambda}_{= \lambda R_\lambda - A R_\lambda}$$

$$\rightsquigarrow R_\lambda A = \lambda R_\lambda - id_{D_A}$$

$$\subset \lambda R_\lambda - id_X = A R_\lambda$$

ii) by part i) $\lambda R_\lambda - id_X = A R_\lambda$

$\mu R_\mu - id_X = A R_\mu$

subtract and get $\mu R_\mu - \lambda R_\mu = A (R_\mu - R_\lambda)$

subtract $\mu (R_\mu - R_\lambda)$ on both sides

$$(\mu - \lambda) R_\lambda = (A - \mu)(R_\mu - R_\lambda)$$

left multiplication by R_μ $(\mu - \lambda) R_\mu R_\lambda = R_\mu - R_\lambda$.

iii) follows from ii) $R_\lambda - R_\mu = (\mu - \lambda) R_\mu R_\lambda$

$(\lambda, \mu) \to (\mu, \lambda)$ $R_\mu - R_\lambda = (\lambda - \mu) R_\lambda R_\mu$

sum and get what you want. ∎

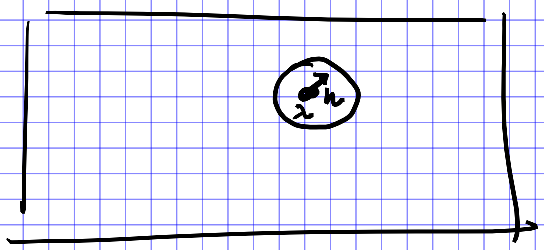
Cor: $\rho(A) \longrightarrow L(X)$

$\lambda \longmapsto (\lambda - A)^{-1} =: R_\lambda \in L(X)$

is actually differentiable.

Pf. (use part ii) above)

$$\frac{R_{\lambda+h} - R_\lambda}{h} \overset{(ii)}{=} - R_{\lambda+h} R_\lambda$$



$$\lim_{h \to 0} \frac{R_{\lambda+h} - R_\lambda}{h} = - \lim_{h \to 0} R_{\lambda+h} R_\lambda = - R_\lambda^2$$

continuity of $z \mapsto R_z$

continuity of multiplication in Banach algebra $L(X)$ ∎