Functional Analysis I – L25 10/12/2020

Structure of the spectrum

$(X, \|\cdot\|_X)$ Banach space$/_{\mathbb{C}}$, $A : D_A \longrightarrow X$ linear.

<u>Assumption</u>: $\Gamma_A \subset X \times X$ <u>closed</u> (this is true e.g. $A \in L(X)$)

$\rho(A) := \{ \lambda \in \mathbb{C} : (\lambda - A) : D_A \longrightarrow X \text{ bijective} \}$.

$\sigma(A) := \mathbb{C} \setminus \rho(A)$.

<u>Def</u>: $\sigma(A) = \sigma_p(A) \sqcup \sigma_c(A) \sqcup \sigma_r(A)$

$\underbrace{\sigma_p(A)}_{\text{point spectrum}} = \{ \lambda \in \mathbb{C} : (\lambda - A) : D_A \longrightarrow X \text{ is } \underline{\text{not}} \text{ injective} \}$

$\underbrace{\sigma_c(A)}_{\text{continuous spectrum}} = \{ \lambda \in \mathbb{C} : (\lambda - A) : D_A \longrightarrow X$ is injective w/ dense image (but not surjective) $\}$

$\underbrace{\sigma_r(A)}_{\text{residual spectrum}} = \{ \lambda \in \mathbb{C} : (\lambda - A) : D_A \longrightarrow X$ is injective w/ non-dense image $\}$

For the <u>point spectrum</u>: $\lambda \in \sigma_p(A)$ is called <u>eigenvalue</u> of A and $\underbrace{E_\lambda}_{\text{eigenspace}} := \operatorname{Ker}(\lambda - A) := \{ x \in D_A : Ax = \lambda x \}$

3 Key examples:

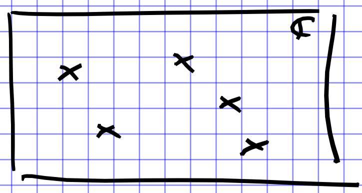
1. $X = \mathbb{C}^n$, $A \in L(X)$ ($n \times n$ complex matrix)

   Given $\lambda \in \mathbb{C}$ TFAE:

   i) $(\lambda - A) : X \circlearrowleft$ injective
   ii) $(\lambda - A) : X \circlearrowleft$ surjective
   iii) $\det(\lambda - A) \neq 0 \in \mathbb{C}$

   $\Longrightarrow \sigma(A) = \sigma_p(A)$ purely <u>point spectrum</u>

Using iii), $1 \le |\sigma(A)| \le n$ points in complex plane.



② (cf. spectral theorem for compact self-adjoint operators...)

H Hilbert space / $\mathbb{C}$, $T \in L(H)$ <u>compact</u> and <u>self-adjoint</u>. <u>Claim 1:</u> $\sigma(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$

<u>why?</u> <u>lemma</u>: if A is a self-adjoint op. then $\lambda \in \sigma(A) \Rightarrow \lambda \in \mathbb{R}$

$$\operatorname{im}(\lambda - T) \underset{\text{orthogonality relation}}{=} \operatorname{Ker}((\lambda - T)^*)^{\perp} \underset{\text{lemma}}{=} \ker(\lambda - T)^{\perp}$$

($(\lambda - T)$ has closed image if $\lambda \neq 0$) $= \lambda(\mathrm{id} - T/\lambda)$)

$\Rightarrow$ $(\lambda - T)$ injective $\Longleftrightarrow$ $(\lambda - T)$ surjective

in part. if $\lambda \in \sigma(T) \setminus \sigma_p(T)$ then $(\lambda - T)$ injective ($\lambda \neq 0$)

hence $(\lambda - T)$ bijective, then $\exists\, (\lambda - T)^{-1} \in L(X)$

thus $\lambda \in \rho(A)$, contradicting $\lambda \in \sigma(T)$. ∎

<u>Claim 2</u> (proven next Thursday):

<u>either</u> T is not injective $\rightsquigarrow$ $0 \in \sigma_p(T)$

<u>or</u> T is injective $\rightsquigarrow$ $\boxed{0 \in \sigma_c(A)}$

③ Right-shift operator $S : \ell^2_{\mathbb{C}} \longrightarrow \ell^2_{\mathbb{C}}$

$$S(a_0, a_1, a_2, \dots) = (0, a_0, a_1, a_2, \dots)$$

Claim: $0 \in \sigma_r(S)$. explicit check for you

$$\operatorname{im}((0-S)) = \operatorname{im}(S) = ((1,0,0,\dots,\dots)^\perp)$$

thus $(0-S)$ not surjective, image not dense.

but 0 is not an eigenvalue of S, indeed

$$Sx = 0 \implies x = 0.$$

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A general fact about the spectrum of bounded operators:

Theorem: Let $A \in L(X)$. Then $\rho(A) \neq \emptyset$, $\sigma(A) \neq \emptyset$.

Also: i) $|z| > r_A := \lim_{n\to\infty} \|A^n\|^{1/n}_{L(X)} \implies z \in \rho(A)$

ii) $r_A = \sup_{z \in \sigma(A)} |z| = \sigma_0$ part I implies $r_A \geq \sigma_0$

Pf. i) (Neumann series) if $|z| > r_A$ set $\tilde{A} = z^{-1} A \in L(X)$

then $r_{\tilde{A}} = \dfrac{r_A}{|z|} < 1$ so $\exists\ (1-\tilde{A})^{-1} \in L(X)$

hence $R_z = (z-A)^{-1} = z^{-1}(1-\tilde{A})^{-1}$

$z \in \rho(A)$ and $= z^{-1}\left(\sum_{n\geq 0} \tilde{A}^n\right)$

$$= z^{-1} + z^{-2}A + z^{-3}A^2 + \dots \in L(X)$$

•) Next: let's check $\sigma(A) \neq \emptyset$. argue by contradiction using Liouville thm. in Complex Analysis.

If $\sigma(A) = \phi$ then $\rho(A) \equiv \mathbb{C} \longrightarrow \mathbb{C}$

$$z \xmapsto{f(z)} \ell(R_z x)$$

for fixed $x \in X$, $\ell \in X^*$

$$\lim_{|z|\to\infty} |f(z)| \leq \underbrace{\left(\|\ell\|_{X^*} \|x\|_X\right)}_{C} \limsup_{|z|\to\infty} \|R_z\|_{L(X)}$$

$$\leq \limsup_{|z|\to\infty} C \underbrace{|z|^{-1}}_{\to 0} \underbrace{\left\|\left(1 - \frac{A}{z}\right)^{-1}\right\|_{L(X)}}_{\to 1} = 0$$

So $f : \mathbb{C} \circlearrowleft$ complex-differentiable hence holomorphic

$$\left(\frac{dR_\lambda}{d\lambda} = -R_\lambda^2\right)$$

and $f \xrightarrow{|z|\to\infty} 0$. Liouville $\Longrightarrow f \equiv$ constant

$\Longrightarrow f \equiv 0 \in \mathbb{C}$.

Thus $\forall x \in X$, $\forall \ell \in X^*$ $\ell(R_z x) = 0$

$\Longrightarrow$ (readily check) $\boxed{R_z = 0}$ but on the other hand $\forall z \in \mathbb{C}$ have $z \in \rho(A)$

so by def. $\boxed{(z - A) R_z x = x}$ $\forall x \in X$

Contradiction!

ii) we already know (by part i)) that $r_A \geq r_0$, so let's prove $r_A \leq r_0$.

Take $\boxed{r > r_A}$ let's note that we can integrate (in the sense of Riemann) functions taking values in any given Banach space, in part. in $\mathcal{L}(X)$. (Bochner integral)

$$\frac{1}{2\pi i}\int_{\partial B_r(0)} z^n \underbrace{(z-A)^{-1}}_{\text{well-defined by i)}} dz$$

$$= \frac{1}{2\pi i}\int_{\partial B_r(0)} z^n \Big(\sum_{k\geq 0} z^{-1-k} A^k\Big) dz$$

$$= \sum_{k\geq 0} A^k \cdot \underbrace{\left(\frac{1}{2\pi i}\int_{\partial B_r(0)} z^{n-k-1} dz\right)}_{\text{not zero iff } k=n} = A^n$$

so we get the representation formula

$$\boxed{r > r_A} \qquad \underbrace{A^n}_{f(A)} = \frac{1}{2\pi i}\int_{\partial B_r(0)} \underbrace{z^n}_{f(z)} (z-A)^{-1} dz$$

$$\Longleftrightarrow \quad \begin{matrix}\forall x\in X \\ \forall \ell \in X^*\end{matrix} \qquad \ell(A^n x) = \frac{1}{2\pi i}\int_{\partial B_r(0)} z^n \ell(R_z x)\, dz$$

Now note that for any fixed $x \in X$ and $\ell \in X^*$

the map $\rho(A) \ni z \longmapsto \ell(R_z x) \in \mathbb{C}$

is <u>holomorphic</u> for $r > \sigma_0 \underset{\text{recall}}{=} \sup_{z\in\sigma(A)} |z|$

thus $\int_{\partial B_r(0)} z^n \ell(R_z x)\, dz$ is a complex number which does <u>not</u> depend on r

as long as $\boxed{r > \sigma_0}$

(homotopic invariance of integral of hol functions) then, we have that (for any fixed $x \in X$, $\ell \in X^*$)

$$\ell(A^n x) = \frac{1}{2\pi i} \int_{\partial B_r(0)} z^n \ell(R_z x) dz$$

for any $\boxed{r > \sigma_0}$ (a priori we had it only for $r > r_A \geq \sigma_0$)

$$\Longrightarrow \quad A^n = \frac{1}{2\pi i} \int_{\partial B_r(0)} z^n (z-A)^{-1} dz \quad \text{for any } r > \sigma_0$$

$$\| A^n \| \leq \frac{1}{2\pi} \underbrace{\left(\sup_{|z|=r} \| R_z \|_{L(X)} \right)} \int_{\partial B_r(0)} |z|^n |dz|$$

$$\| A^n \| \leq r^{n+1} M(r)$$

$$\| A^n \|^{1/n} \leq r^{\left(1+\frac{1}{n}\right)} M(r)^{1/n}$$

so, if we let $n \to \infty$ $\boxed{\begin{array}{c} r_A \leq r \\ (\forall r > \sigma_0) \end{array}}$ $\Longrightarrow r_A \leq \sigma_0$ ∎

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Some spectral calculus

Setting as above, $A \in L(X)$

assume $\sigma(A) \subset \Omega$ open and bounded

More generally $\sigma(A) \subset \bigcup_{1 \leq \ell \leq L} \Omega_\ell$

w/ Ω_ℓ open and bounded (say w/ C^1 boundary $\partial\Omega_\ell = \gamma_\ell$).

Suppose f is a "reasonably simple function", say

$$f(z) = p(z) + \sum_{m=1}^{M} c_m(\alpha_m - z)^{-k_m}$$

w/ $p \in \mathbb{C}[z]$

$c_m \in \mathbb{C}$, $\alpha_m \in \rho(A)$, $k_m \in \mathbb{N}$ $\forall m$.

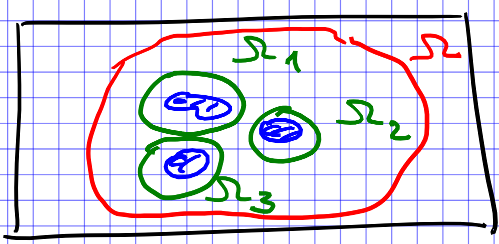
(*) $(\alpha_m \notin \Omega)$

Then we can give a meaning to

$$f(A) := \underbrace{p(A) + \sum_{m=1}^{M} c_m \underbrace{(\alpha_m - A)^{-k_m}}_{R_{\alpha_m}^{k_m}}}_{\in L(X)}$$

→ Q. what is $\sigma(f(A))$ in terms of A and f?

Answer:



$f \in C^0(\Omega, \mathbb{C})$

$\Omega_i \subsetneq \Omega \quad i = 1, \dots, \ell$

we agree that $\displaystyle\int_\gamma f\,dz = \sum_{1 \le \ell \le L} \int_{\gamma_\ell} f\,dz$

Lemma: $\forall n \in \mathbb{N}$ $\displaystyle A^n = \frac{1}{2\pi i}\int_\gamma z^n R_z\,dz$

Pf: true if $L = 1$ and $\mathrm{spt}(\gamma) = \partial B_r(0)$ for $r > r_A$ (by what we have seen in the proof of the previous thm). In general (picture as above) apply Cauchy's theorem (appeal to homological invariance ...). ∎

Lemma (extension of the above)

a) $\alpha \notin \Omega$ $\quad \forall n \in \mathbb{Z}$ $\quad (\alpha - A)^n = \frac{1}{2\pi i}\int_\gamma (\alpha - z)^n R_z \, dz$

$(\alpha \in \rho(A))$ $\qquad =: y_n$

(b) if $n \in \mathbb{N}$ (i.e. $n \geq 0$) then this formula ok $\forall \alpha \in \mathbb{C}$.

Pf: for $n = 0$ this coincides w/ lemma above.

General case done by recursion: if $z \in \rho(A)$ then R_z is well-defined and $(\alpha - A) R_z = 1 + (\alpha - z) R_z$

$$\Rightarrow (\alpha - A) R_z (\alpha - z)^n = (\alpha - z)^n + (\alpha - z)^{n+1} R_z$$

take this identity and integrate $\frac{1}{2\pi i}\int_\gamma$

$$\boxed{(\alpha - A)\, y_n = \frac{1}{2\pi i}\int_\gamma (\alpha - z)^n \, dz + y_{n+1}}$$

Given y_n can solve for y_{n+1} (always, under no hp. on α)

Given y_{n+1} can solve for y_n if $\alpha \in \rho(A)$ $\blacksquare$

Moral $\quad f(A) = \frac{1}{2\pi i}\int_\gamma f(z) R_z \, dz$

for f satisfy $(*)$ $\qquad$ We can now answer the question:

Theorem (Spektralabbildungssatz) Let f satisfy $(*)$. Then:

i) $f(A) \in GL(X) \iff \forall z \in \sigma(A)\ f(z) \neq 0$

ii) $\sigma(f(A)) = f(\sigma(A))$.

Pf: $\boxed{\Longleftarrow}$ if $f(z) \neq 0$ on any $z \in \sigma(A)$ pick $\Omega_1 \subset \Omega$ an open regular domain w/ (w/ C^1 bdry) $\sigma(A) \subset \Omega_1$ where f does not vanish

Now $g(z) = \dfrac{1}{f(z)}$

$g: \Omega_1 \to \mathbb{C}$ holomorphic (in Ω_1).

thus (lemma, $n=0$)

$$\underset{\text{identity map}}{1} = \frac{1}{2\pi i} \int_{\partial\Omega_1} 1\, R_z\, dz = \frac{1}{2\pi i} \int_{\partial\Omega_1} f(z) g(z) R_z\, dz$$

$$= f(A)\, g(A)$$

similarly $1 = g(A) f(A)$ $\rightsquigarrow f(A) \in GL(X)$.

$\boxed{\Longrightarrow}$ $f(\lambda) = 0$ for some $\lambda \in \sigma(A)$ $f(z) = g(z)(z-\lambda)$

holomorphic in Ω

$$\rightsquigarrow f(A) \overset{(1)}{=} g(A)(A-\lambda) \overset{(2)}{=} (A-\lambda) g(A)$$

if $(A-\lambda)$ not injective $\overset{(1)}{\rightsquigarrow}$ $f(A)$ not injective

if $(A-\lambda)$ not surjective $\overset{(2)}{\rightsquigarrow}$ $f(A)$ not surjective

either way $f(A) \notin GL(X)$.

ii) given any $\beta \in \mathbb{C}$ by i) applied to $h(z) := f(z) - \beta$ have

$$g(A) = f(A) - \beta \notin GL(X) \iff \exists\, z \in \sigma(A)\ \ g(z) = 0 = f(z) - \beta$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$\beta \in \sigma(f(A)) \overset{!}{\iff} \beta \in f(\sigma(A))$$

∎