Functional Analysis I — L26 14/12/2020

Spectral Theory in Hilbert Spaces

↳ symmetric $A \subset A^*$ ✓

self-adjoint $A = A^*$

(the two things coincide if $A \in \mathcal{L}(H)$)

Setup: $(H, \underbrace{(\,,\,)}_{})$ Hilbert space / $\mathbb{C}$

recall ↖ Hermitian product $(x,y) = \overline{(y,x)}$ $\forall x,y \in H$

$A : D_A \subset H \longrightarrow H$ linear, $\overline{D}_A = H$ (densely defined)

Prop.: Let A be symmetric, then its eigenvalues are real

i.e. $\sigma_p(A) \subset \mathbb{R}$.

Pf: by def $\lambda \in \sigma_p(A)$ if $\exists x \in D_A \setminus \{0\}$ $Ax = \lambda x$.

$$\underbrace{\lambda \|x\|^2}_{} = (Ax, x) \underset{A \subset A^*}{=} (A^* x, x) \underset{\text{fund. identity}}{=} (x, Ax) = \overline{(Ax, x)} = \underbrace{\overline{\lambda} \|x\|^2}_{}$$

$\Longrightarrow \lambda = \overline{\lambda} \iff \lambda \in \mathbb{R}$ ∎

General question: if A is symmetric, is it true that $\sigma(A) \subset \mathbb{R}$?

What if A is self-adjoint?

Example: a symmetric operator w/ non-real spectrum

$$H = L^2((0,1); \mathbb{C})$$

$$A := i \frac{d}{dt} : D_A \longrightarrow H$$

$$D_A := \{ f \in C^1([0,1]; \mathbb{C}), \underbrace{f(0) = f(1) = 0}_{} \}$$

Claim 1 : $A \subset A^*$ i.e. A is symmetric

pick $f \in D_A$ must check that
$$\begin{array}{ccc} u & \longmapsto & (f, Au) \\ \cap & & \\ D_A & \longrightarrow & \mathbb{C} \end{array}$$
is bounded.

$$(f, Au) = \int_0^1 f(t)\, \overline{i u'(t)}\, dt = -i \int_0^1 f(t)\, \overline{u'(t)}\, dt$$

$$= -i \left[\cancel{f \cdot \bar{u} \Big|_0^1} - \int_0^1 f'(t)\, \overline{u(t)}\, dt \right]$$

$$= +i \int_0^1 f'(u)\, \overline{u(t)}\, dt$$

$$|(f, Au)| \leq \|f'\|_{L^2} \|u\|_{L^2}$$
$$\leq \|f\|_{C^1} \|u\|_{L^2}$$

so $f \in D_A \implies f \in D_{A^*}$ and $A^* f = i f'$

Claim 2 : (spectrum of A) i) $\sigma_p(A) = \emptyset$

ii) $\sigma(A) = \mathbb{C}$

i) $(\lambda - A) u = 0 \iff u' = -i\lambda u$

(wlog $\lambda \in \mathbb{R}$) general sol. $u(t) = a\, e^{-i\lambda t}$

but $u \in D_A \implies u(0) = u(1) = 0 \underset{\text{direct check}}{\implies} a = 0$

So no eigenvalues!

ii) enough to prove: $\forall \lambda \in \mathbb{C}$ the map

$(\lambda - A) : D_A \longrightarrow H$ is not surjective

e.g. $g(s) = e^{-i\lambda s} \notin \operatorname{im}(\lambda - A)$

why? $(\lambda - A)u = g \iff u' = -i\lambda u + ig$

so $u(t) = \alpha e^{-i\lambda t} + i\int_0^t e^{i\lambda(s-t)} g(s)\,ds$

$$\underset{\text{choice of } g}{=} \alpha e^{-i\lambda t} + i e^{-i\lambda t} t$$

now $u \in D_A$ forces $u(0) = 0 \implies \alpha = 0$

$u(1) = 0 \implies \underbrace{e^{-i\lambda} = 0}_{\text{impossible}}$.

Bonus question for you: determine $\sigma_c(A)$ and $\sigma_r(A)$ and draw them on the complex plane.

Things are much better for self-adjoint operators. We first need a preliminary result: we have seen for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ then $(\lambda - A) : D_A \to H$ is injective as soon as A is symm.

Lemma (stronger form): let A be symmetric. Given $z \in \mathbb{C}$

$\forall u \in D_A$: $\|(z-A)u\| \geq |\mathrm{Im}(z)|\,\|u\|$

and if $z - A : D_A \to H$ is surjective then $(z-A)^{-1} \in GL(H)$

w/ estimate $\|(z-A)^{-1}\|_{\mathcal{L}(H)} \leq |\mathrm{Im}(z)|^{-1}$ ∎

Pf. $\forall u \in D_A$ $(u, Au) = (Au, u)$

$$= \overline{(u, Au)}$$

So $\begin{cases} (u, Au) \in \mathbb{R} \\ (Au, u) \in \mathbb{R} \end{cases}$ thus I can write the following chain:

$|\mathrm{Im}\,(u, zu)| = |\mathrm{Im}\,(u, (z-A)u)| \leq |(u, (z-A)u)|$

$= |\mathrm{Im}\, z \cdot (u,u)| = |\mathrm{Im}(z)|\,\|u\|^2 \qquad \leq \|u\|\,\|(z-A)u\|$

Simplify $\|u\|$ and get $|\mathrm{Im}(z)|\,\|u\| \leq \|(z-A)u\|$ ∎

Prop. (Spectral characterisation of self-adjointness)

Setting above, let $A : D_A \subset H \longrightarrow H$ be symmetric. TFAE:

i) $A = A^*$ (i.e. A is actually self-adjoint)

ii) $\sigma(A) \subset \mathbb{R}$

iii) $\exists\ z_1, z_2 \in \rho(A)$ w/ $\mathrm{Im}(z_1) < 0 < \mathrm{Im}(z_2)$.

Achtung! if $A \in L(H)$ then $A^* \in L(H)$,

thus "symmetric $\Longleftrightarrow$ self-adjoint".

In that case (i.e. if $A \in L(H)$ and symmetric) then

i) is TRUE, hence ii) and iii) are TRUE as well

$\sigma(A) \subset \mathbb{R}$

we already know a lot more: the complement of $D(\underline{0}, r_A)$ is contained in $\rho(A)$.

Pf. iii) $\Rightarrow$ ii) $\Longrightarrow$ i) $\Longrightarrow$ iii)

$\boxed{\text{iii}) \Rightarrow \text{ii}}$

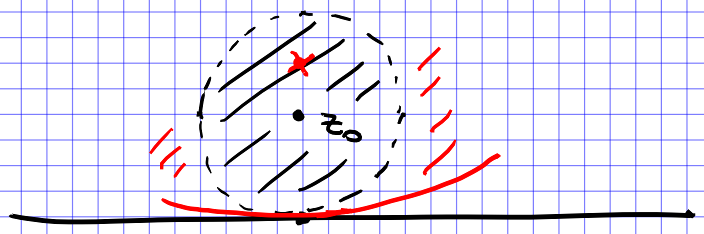
given $z_0 \in \rho(A)$, $\mathrm{Im}(z_0) > 0$ — we'll now show $\Longrightarrow \rho(A) \supset \{\mathrm{Im}(z) > 0\}$

By previous lemma: if $z_0 \in \rho(A)$ then $\|R_{z_0}\| \leq \dfrac{1}{|\mathrm{Im}(z_0)|}$

Thus $D := \{ z \in \mathbb{C} : |z - z_0| < |\mathrm{Im}(z_0)| \}$

$\subset \{ z \in \mathbb{C} : |z - z_0| < \|R_{z_0}\|^{-1} \} \subset \rho(A)$

last time



proceed by covering scheme

given $z_i \in \rho(A) \rightsquigarrow z_{i+1} = \mathrm{Re}(z_i) + \frac{3}{2}\,\mathrm{Im}(z_i)$

$(\mathrm{Im}(z_i) > 0)$

Proceed till you cover $\{\mathrm{Im}(z) > 0\}$.

$\boxed{ii) \Rightarrow i)}$ Since $A \subset A^*$ by hp. must show $D_{A^*} \subset D_A$.

So let $u \in D_{A^*}$, by $\sigma(A) \subset \mathbb{R}$ have $\begin{cases} +i \in \rho(A) \\ -i \in \rho(A) \end{cases}$

i.e. $\exists\, v \in D_A \quad (A - i)v = \underbrace{(A^* - i)u}_{\in H}$

$(A^* - i)v \underset{D_A \subset D_{A^*}}{=}$

So $u - v \in \mathrm{Ker}(A^* - i)$.

Then pick $w \in D_A$ s.t. $(A + i)w = \underbrace{u - v}$

$\|u - v\|^2 = (u - v, (A + i)w) = ((A^* - i)(u - v), w)$
$= 0$

$\Rightarrow \underset{\in D_{A^*}}{u} = \underset{\in D_A}{v} \rightsquigarrow u \in D_A$

$\boxed{i) \Rightarrow iii)}$ We'll prove that: if $A = A^* \Rightarrow \boxed{\begin{matrix} z \in \mathbb{C} \\ \mathrm{Im}(z) \neq 0 \end{matrix} \Rightarrow z \in \rho(A)}$

by the Lemma, given $z \in \mathbb{C}$ w/ $\mathrm{Im}(z) \neq 0$ it's enough to check surjectivity of $z - A : D_A \longrightarrow H$.

1^{st} : image is closed. $v_k = (z-A)u_k \quad u_k \in D_A, \quad k \in \mathbb{N}$

Coercivity estimate: $\| u_k - u_e \| \leq \dfrac{1}{|\mathrm{Im}(z)|} \| v_k - v_e \|$

Know $v_k \longrightarrow v \implies (v_n)$ Cauchy $\implies (u_n)$ Cauchy

(H complete) $u_n \xrightarrow{\text{in } H} u$

but $\underset{\text{(has closed graph!)}}{A = A^*} \implies u \in D_A$

$(z-A)u = \lim_{k \to \infty} (z-A)u_k$

$= \lim_{k \to \infty} v_k$

$= v$

2^{nd} : image is all. $M = \text{image}(z-A)$

by contradiction $M \subsetneq H$

Then $v \in M^{\perp} \subset H$ i.e. $\begin{cases} (v, (z-A)u) = 0 \\ \forall u \in D_A \end{cases}$

$\boxed{\|v\| = 1}$

hence $\underbrace{(v, Au) = \bar{z}(v, u)}$

by this equation $v \in D_{A^*} = D_A$

hence (by key identity)

$\begin{cases} (A^* v, u) = (v, Au) = \bar{z}(v, u) \\ \forall u \in D_A \end{cases}$ dense in H

$\longrightarrow A^* v = \bar{z} v$ If I apply Lemma above to $\bar{z}$

$|\mathrm{Im}(z)| \, \|v\| \leq \|(\bar{z} - A)v\| = \|\bar{z}v - Av\|$

$= \|\bar{z}v - A^* v\| = \|\bar{z}v - \bar{z}v\| = 0$

$\Rightarrow \boxed{v = 0}$

To get to the spectral theorem we'll now introduce some definitions and examine one key example:

<u>Def</u>: In the setting above, let $T \in L(H)$. Then:

- T is <u>self-adjoint</u> if $T = T^*$
- T is <u>normal</u> if $TT^* = T^*T$
- T is <u>unitary</u> if $T^* = T^{-1}$

<u>Rmk</u>.

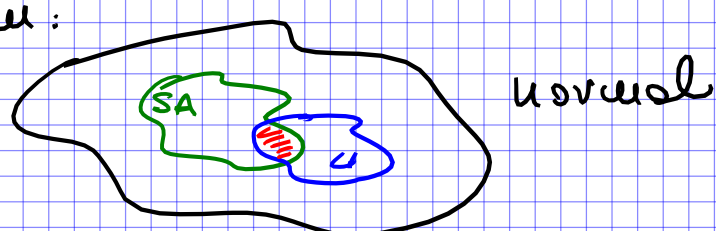
1) $T \in L(H)$ self-adjoint $\Rightarrow$ T normal

$$TT^* = T^2 = T^*T.$$

2) $T \in L(H)$ unitary $\Longrightarrow$ T normal

$$TT^* = id = T^*T$$

A Venn diagram:



<u>Rmk</u>. (A characterisation of normal operators)

$$T \text{ is normal} \iff \underbrace{\|Tx\| = \|T^*x\| \quad \forall x \in H}$$

in part. this is true for self-adjoint or unitary operators

<u>why</u>? $\boxed{\Rightarrow}$ standard identities

$$\mathbb{R} \ni \|Tx\|^2 = (Tx, Tx) = (T^*Tx, x)$$

$$\mathbb{R} \ni \|T^*x\|^2 = (T^*x, T^*x) = (x, TT^*x) = (TT^*x, x)$$

$\boxed{\Leftarrow}$ by identities above, have that

$$(*) \quad (TT^*x, x) = (T^*Tx, x) \quad \forall x \in H$$

a) apply $(*)$ to $x+y$ $\quad (x, y \in H)$

$$2 \operatorname{Re} (T^*Tx, y) = 2 \operatorname{Re} (TT^*x, y) \quad \forall x, y \in H$$

b) apply $(*)$ to $x+iy$ $\quad (x, y \in H)$

$$2 \operatorname{Im} (T^*Tx, y) = 2 \operatorname{Im} (TT^*x, y) \quad \forall x, y \in H$$

Combine a) and b) to get

$$(T^*Tx, y) = (TT^*x, y) \quad \forall x, y \in H$$

$$\Longrightarrow \quad T^*Tx = TT^*x \quad \forall x \in H \iff TT^* = T^*T \quad \blacksquare$$

<u>Norm of normal operators</u>:

<u>Prop.</u> If $T \in \mathcal{L}(H)$ normal, then

already proven

$$\|T\|_{\mathcal{L}(H)} = \sup_{\lambda \in \sigma(T)} |\lambda| = r_T$$

<u>Pf.</u> enough to check $\boxed{\|T^n\| = \|T\|^n}$ $\forall n \in \mathbb{N}_*$

induction on n $\qquad \boxed{n = 1}$ trivial.

$\boxed{n \Rightarrow n+1}$ $\quad T$ normal $\Longrightarrow \|Tz\| = \|T^*z\| \; \forall z \in H$

Given $x \in H$, $\|x\| = 1$

$$\|T^n x\|^2 = (T^n x, T^n x) = (T^{n-1}x, T^*T^n x)$$

$$\leq \|T^{n-1}x\| \, \|T^*(T^n x)\| \overset{(=)}{\leq} \|T^{n-1}x\| \, \|T^{n+1}x\|$$

rmk.

$$\leq \|T^{n-1}\| \, \|T^{n+1}\|$$

so we proved $\| T^n \|^2 \leq \| T^{n-1} \| \| T^{n+1} \|$

$$\| T \|^{2n} \overset{\text{induction}}{=} \| T^n \|^2 \leq \| T^{n-1} \| \| T^{n+1} \| \leq \| T \|^{n-1} \| T^{n+1} \|$$

simplify and get $\| T \|^{n+1} \leq \| T^{n+1} \| \underset{\text{general}}{\leq} \| T \|^{n+1}$

$$\Longrightarrow \| T \|^{n+1} = \| T^{n+1} \| \quad \blacksquare$$

We'll use this fact next time in proving the spectral theorem for (compact) self-adjoint operators.