Functional Analysis I - L27 17/12/2020

Spectral theorem for compact, self-adjoint operators

Example: (later we'll this example describes the general situation...)

$H = \ell^2_{\mathbb{C}}$, fix a sequence $(\lambda_k)_{k\in\mathbb{N}} \in \ell^\infty_{\mathbb{C}}$ ✓

take $T: \ell^2_{\mathbb{C}} \circlearrowleft$ defined by

$$T(a_0, a_1, a_2, \dots) = (\lambda_0 a_0, \lambda_1 a_1, \lambda_2 a_2, \dots)$$

- bounded $\|Ta\|_{\ell^2} \le \underbrace{\|\lambda\|_{\ell^\infty}} \|a\|_{\ell^2}$
- adjoint (?) $(Ta, b) = (a, T^* b)$

  recall: we're working / $\mathbb{C}$ $(y,z) = \overline{(z,y)}$

  $$T^*(a_0, a_1, \dots) = (\bar{\lambda}_0 a_0, \bar{\lambda}_1 a_1, \dots)$$
- self-adjoint (?) $\boxed{T = T^* \iff \lambda_k \in \mathbb{R} \ \forall k \in \mathbb{N}}$
- normal (?) $TT^* = T^*T$ you can check this is always the case

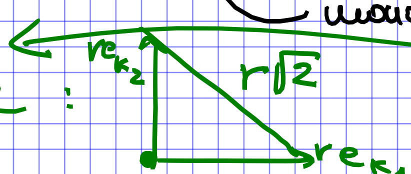
  $$TT^* a = T^*T a = (|\lambda_0|^2 a_0, |\lambda_1|^2 a_1, \dots)$$
- unitary (?) $TT^* = T^*T = id \iff |\lambda_k| = 1 \ \forall k \in \mathbb{N}$
- compact (?) T is compact $\iff \lambda_k \xrightarrow{k\to\infty} 0$

$\boxed{\Rightarrow}$ by contradiction, assume $\exists \Lambda \subset \mathbb{N}$ unbounded set w/ $\forall k \in \Lambda$ have $|\lambda_k| \ge r > 0$

Then, if so, $T(B_1(0, \ell^2_{\mathbb{C}})) \supset \underbrace{B_r(0; Y)}$

where $Y = \text{span}_{\mathbb{C}}(e_k : k \in \Lambda)$ e_k monomial sequence

not compact

(argument as in Chapter 2: 

⇝ no Cauchy subseq)

$\boxed{\Leftarrow}$

notation: x^j_k k-th element of a sequence x^j

i.e. $\underbrace{x^j}_{\in \ell^2_{\mathbb{C}}} = (x^j_k)_{k\in\mathbb{N}}$

1) take a sequence (x^j) in $\ell^2_{\mathbb{C}}$

w/ $|x^j| \leq 1$

set $Tx^j =: y^j$ we want to extract a strongly converging subsequence in $\ell^2_{\mathbb{C}}$

by Eberlein–Smulyan $\exists\ x \in \ell^2_{\mathbb{C}}$ $x^j \xrightarrow[j\to\infty]{w} x$

(up to a subsequence, but I'll not rename)

in part. for any fixed $k_0 > 0$ have $x^j_k \longrightarrow x_k$

$0 \leq k \leq k_0$

2) given $\epsilon > 0$ $\exists$ $\ell = \ell(\epsilon)$

w/ $|\lambda_k| < \epsilon$ $\forall k \geq \ell$

set $y = Tx$ we claim $\boxed{y^j \xrightarrow{\ell^2_{\mathbb{C}}} y}$

$y^j = Tx^j$

why is this true? $y^j_{(\ell)} = (\underbrace{0, \dots, 0}_{\ell-1 \text{ slots}}, y^j_\ell, y^j_{\ell+1}, \dots)$

$$\limsup_{j\to\infty} \underbrace{\|y^j - y\|^2}_{\text{use Parseval}} = \limsup_{j\to\infty} \underbrace{\|y^j_{(\ell)} - y_{(\ell)}\|^2}$$

(and pointwise convergence of the first ℓ components)

$$\leq \limsup_{j\to\infty} \epsilon^2 \underbrace{\|x^j - x\|^2}_{\leq 4}$$

$$\leq 4\epsilon^2$$

Same argument is true $\forall \epsilon > 0$ $\rightsquigarrow$ $\limsup\limits_{j\to\infty} \|y^j - y\|^2 = 0$

i.e. $y^j \longrightarrow y$ ∎

Spectrum of this operator: $\sigma(T) = \overline{\{\lambda_k : k \in \mathbb{N}\}}$

- λ_k eigenvalue of T w/ eigenvector $e_k = (0, \dots, 0, 1, 0, \dots)$
  $(\lambda_k \in \sigma_p(T) \subset \sigma(T))$

  these are the only eigenvalues, because if $Tx = \lambda x$ get
  $(\lambda_0 x_0, \lambda_1 x_1, \dots) = (\lambda x_0, \lambda x_1, \dots)$
  if $x \neq 0$ then $\lambda = \lambda_i$ for some i
- if $\lambda \neq \lambda_k$ $(\forall k)$ then $(\lambda - T) : \ell^2_{\mathbb{C}} \circlearrowleft$ injective
- how about surjectivity? given $b \in \ell^2_{\mathbb{C}}$ want to solve
  $(\lambda - T)a = b \rightsquigarrow a_k = \dfrac{b_k}{\lambda - \lambda_k}$ but we don't know if $a \in \ell^2_{\mathbb{C}}$ in case $\lambda_k \to \lambda$

Reality check: the linear inverse $(\lambda - T)^{-1}$ is well-defined and bounded iff $\lambda \notin \overline{\{\lambda_k : k \in \mathbb{N}\}}$

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Spectral Theorem: let $\begin{cases} H \text{ be Hilbert} / \mathbb{C} \text{ and let} \\ \dim_{\mathbb{C}} H = \infty \end{cases}$

$T \neq 0$ self-adjoint and compact. Then there are at most countably many non-zero eigenvalues $\lambda_k \in \mathbb{R} \setminus \{0\}$.
Also $\lambda_k \to 0$ as $k \to \infty$ and

(a) $H = \operatorname{Ker}(T) \oplus \overline{\operatorname{span}\{e_k : k \in \mathbb{N}\}}$

(b) $Tx = \sum_{k \in \mathbb{N}} \lambda_k e_k (x, e_k) \quad \forall x \in H.$

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Moral : setting above, T self-adjoint and compact $\rightsquigarrow$ T as an infinite diagonal matrix

Rmk. 1 : eigenvalues may have multiplicity

e.g. $\lambda_{10} = \lambda_{11} = \lambda_{12}$

Rmk. 2 : H is not required to be separable, and (in gen.) $\ker(T)$ may be a closed, non-separable subspace of H. However, if H is separable, then $\ker(T)$ is also a separable Hilbert space in its own so (in this case) I can find an Hilbertian basis for H diagonalising T.

Proof. we already know:

① $\sigma(T) \subset \mathbb{R} \subset \mathbb{C}$ (spectrum is real) [T self-adjoint]

② T compact and self-adjoint $\Longrightarrow$ $\sigma(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$

(orthogonality relations)

③ T normal $\Longrightarrow$ set $r_0 := \|T\|_{L(X)}$

$\sigma(T) \subset \overline{B_{r_0}(\underline{0})}$

Step 1: given any $\lambda \in \sigma_p(T) \setminus \{0\}$ consider $X_\lambda := \ker(\lambda - T)$. We know $\dim(X_\lambda) < \infty$ (proof already seen). Also (Linear Algebra argument)

$\lambda_k \neq \lambda_e \quad \rightsquigarrow X_{\lambda_k} \perp X_{\lambda_e}$

$Te_k = \lambda_k e_k, \; Te_e = \lambda_e e_e \quad \lambda_k (e_k, e_e) = (Te_k, e_e) = (e_k, Te_e)$
$= (e_k, \lambda_e e_e) = \lambda_e (e_k, e_e)$

Step 2: "$0 \in \mathbb{C}$ is the only potential accumulation point for $\sigma(T)$"

i.e. $\forall r > 0 \quad \sigma_p(T) \setminus B_r(0)$ is <u>finite</u>.

<u>Pf</u> (by contradiction) if not, then (possibly extracting a subsequence...) $\lambda_k \longrightarrow \lambda \neq 0 \in \mathbb{C}$

<u>wlog</u> $\lambda_k \neq \lambda_\ell$ if $k \neq \ell$

$\forall k$ let e_k be an eigenvector corresponding to λ_k

$\|e_k\| = 1$

<u>T compact</u> $\rightsquigarrow$ $Te_k \longrightarrow y \in H$ (possibly extracting a 2nd subsequence)

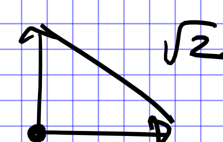
but then

$$e_k = \underset{\downarrow}{\lambda_k^{-1}} \; \underset{\downarrow}{\underbrace{Te_k}}$$
$$\qquad \lambda^{-1} \qquad y$$

$\rightsquigarrow$ (e_k) converging

$\Downarrow$

(e_k) Cauchy



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hence

$$\underbrace{\sigma_p(T) \setminus \{0\}}_{\text{this is countable!}} = \bigcup_{k \in \mathbb{N}} \sigma_p(T) \setminus B_{1/k}(0)$$

Step 3: using the notation as above (e_k) is trivially an Hilbertian basis

$$X := \overline{\operatorname{span}\{e_k : k \in \mathbb{N}\}} = \overline{\bigoplus_{\lambda \in \sigma_p(T) \setminus \{0\}} X_\lambda}$$

closed subspace of H (itself a Hilbert space)

$\rightsquigarrow$ $\forall x \in X$ have $x = \sum_{k \in \mathbb{N}} (x, e_k) e_k$

action of T on X: $\quad Tx = \sum_{k \in \mathbb{N}} (x, e_k) Te_k = \sum_{k \in \mathbb{N}} \lambda_k (x, e_k) e_k.$

<u>Step 4</u>: set $Y := X^{\perp} \subset H$ then

$$\boxed{\underline{\text{Claim}}: \quad Y = \ker(T)}$$

(and so $\quad H = X \oplus X^{\perp} = \ker(T) \oplus \overline{\operatorname{span}\{e_k : k \in \mathbb{N}\}}$

- $TY \subseteq Y \qquad \Big((e_k, Ty) = (Te_k, y) = \lambda_k \underbrace{(e_k, y)}_{=0} = 0$

set $\quad T_0 = T\big|_Y \in L(Y)$

- $\cdot\,\cdot\quad T_0$ <u>compact</u> and <u>self-adjoint</u>

$\rightsquigarrow \quad \sigma(T_0) \subset \sigma_p(T_0) \cup \{0\}$

<u>but</u> let $\lambda_0 \in \sigma_p(T_0)$ then $T_0 e_0 = \lambda_0 e_0 = Te_0$

$\rightsquigarrow$ either $\lambda_0 = 0$ <u>or</u> $\underbrace{e_0 \in X \cap Y = \{0\}}_{\text{contrary to the def. of eigenvector}}$

$\Longrightarrow \underbrace{\sigma(T_0)}_{\text{cannot be empty!}} = \{0\} \quad \Longrightarrow \quad \|T_0\| = 0$

(char of spectral radius of normal operators) $\quad \Longrightarrow \quad Y = \ker(T)$ $\blacksquare$

<u>Rmk.</u>

- if $\ker(T) \neq 0$ then $0 \in \sigma_p(T)$
- if $\ker(T) = 0$ then $X = H$ (so H is <u>separable</u>)

and $0 \in \sigma_c(T)$ if $\dim_{\mathbb{C}} H = \infty$.

Extra 1 : Courant-Fischer characterization of eigenvalues

Setting above (same as in the statement of the theorem)

assume further that T be positive definite i.e.

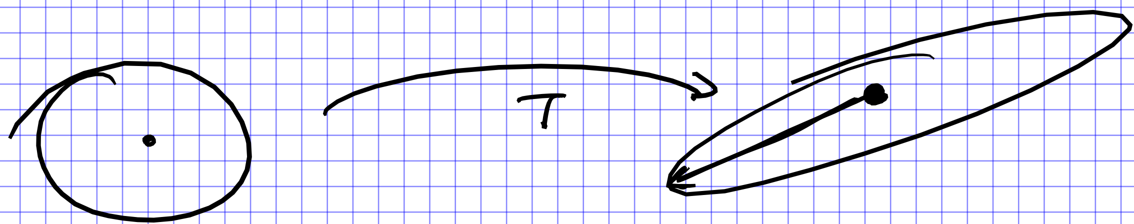
$$\forall x \in H \setminus \{0\} \qquad (Tx, x) > 0$$

Then we can order the eigenvalues

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq \dots \to 0$$

$$\lambda_k = \underbrace{\sup_{\substack{M \subset H \\ \dim M \geq k}} \inf_{\substack{x \in M \\ \|x\|=1}} (Tx, x)} =: \mu_k$$

e.g. $\quad \lambda_1 = \sup_{\|x\|=1} (Tx, x)$



Pf. $\boxed{\lambda_k \leq \mu_k}$ pick $M_k := \operatorname{span}\{e_j : 1 \leq j \leq k\}$

$$\mu_k \geq \inf_{\substack{x \in M_k \\ \|x\|=1}} (Tx, x) = \inf_{\substack{x \in M_k \\ \|x\|=1}} \Big(T\Big(\sum_{1\leq j\leq k} (x,e_j)e_j\Big), \sum_{1 \leq j \leq k} (x,e_j)e_j\Big)$$

$$= \inf_{\substack{x \in M_k \\ \|x\|=1}} \sum_{1 \leq i \leq k} (x, e_i)^2 \lambda_i \geq \lambda_k \|x\|^2 = \lambda_k$$

$\boxed{\lambda_k \geq \mu_k}$ There are two cases here :

① $\exists x_0 \in M, \|x_0\| = 1 \qquad x_0 \perp M_k$

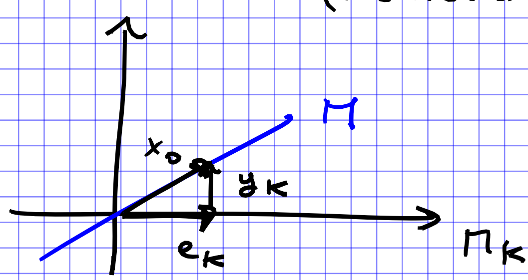
$$(Tx_0, x_0) = \left(T \sum_{i \geq 0} (x_0, e_i) e_i, \sum_{j \geq 0} (x_0, e_j) e_j\right)$$

$$= \left(\sum_{i > k} \lambda_i (x_0, e_i) e_i, \sum_{j > k} (x_0, e_j) e_j\right)$$

$$= \sum_{i > k} \lambda_i (x_0, e_i)^2 \leq \lambda_k \|x_0\|^2 = \lambda_k$$

(2) else, $\pi_k : M \longrightarrow M_k$ is a linear bijection

(restriction of projection $H \to M_k$)



set $x_0 := \pi_k^{-1} e_k = e_k + y_k$

$$(Tx_0, x_0) = (Te_k + Ty_k, e_k + y_k)$$

$$= (Te_k, e_k) + (Ty_k, y_k)$$

$$\leq \lambda_k + \underbrace{\lambda_k \|y_k\|^2}_{\text{as in (1)}} \leq \lambda_k \|x_0\|^2 = \lambda_k$$

$\forall M \rightsquigarrow \boxed{\mu_k \leq \lambda_k}$ ■

9. how about spectral theory / $\mathbb{R}$?

a) the $\mathbb{R}$-case is "harder to investigate"
(e.g. no analytic continuation...)

b) definite link between $\sigma_{\mathbb{R}}$ and $\sigma_{\mathbb{C}}$
(see short note)

c) however, the spectral theorem above is TRUE in the very same form if H Hilbert space / $\mathbb{R}$ and $T : H \circlearrowleft$ linear, self-adjoint compact op.
(no need to change even a single word in the proof!)

In FA 2 we'll e.g. see can "diagonalize" the Laplace operator, i.e.

$$\Delta : H^1_0(\Omega) \cap H^2(\Omega) \longrightarrow L^2(\Omega)$$ $\Omega \subset \mathbb{R}^n$ open, bounded smooth $\partial\Omega$

$\exists$ o.n. basis of L^2, $(\varphi_k)_{k \geq 0}$

and eigenvalues $(\lambda_k)_{k \geq 0}$

w/ $\Delta \varphi_k = - \lambda_k \varphi_k$

note: $-\Delta$ is positive definite

$$(-\Delta u, u) = \int_\Omega |\nabla u|^2 dx \geq 0$$

and thus (λ_k) are characterised by Courant-Fischer.