ETH Zurich HS 2020
D-MATH, D-PHYS
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Coordinator
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Probability Theory

Exercise Sheet 1

Exercise 1.1 Consider the Probability space $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \mathbb{P})$ where $\mathbb{P}(dx) = \frac{1}{2\pi} \exp\{-\frac{1}{2}(x_1^2 + x_2^2)\}\, dx$ with $x = (x_1, x_2)$ for $x \in \mathbb{R}^2$ and dx the Lebesgue measure on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$. Find the distribution of the random variable

$$Z : x = (x_1, x_2) \in \mathbb{R}^2 \mapsto x_1^2 + x_2^2 \in \mathbb{R}.$$

Exercise 1.2 Let $\mathcal{Z} := (A_i)_{i \in I}$ be a countable decomposition of a set $\Omega \neq \emptyset$ in "atoms" A_i, that is $\Omega = \bigcup_{i \in I} A_i$, where $A_i \cap A_k = \emptyset$ for $i \neq k$, and I countable.

(a) Show that the σ-algebra generated by $\mathcal{Z}$ is of the form

$$\sigma(\mathcal{Z}) = \left\{ \bigcup_{i \in J} A_i \,\middle|\, J \subseteq I \right\}.$$

Hint: Recall the definition of $\sigma(\mathcal{Z})$.

(b) Show that the family of $\sigma(\mathcal{Z})$-measurable random variables is exactly the family of functions on Ω that are constant on "atoms" (that is, all functions f such that for each i, f is constant on A_i).

Exercise 1.3 Let Ω be a non-empty set and let $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ be two functions. The σ-algebra on Ω generated by X is defined by $\sigma(X) := \left\{ X^{-1}(B) \mid B \in \mathcal{R} \right\}$, where $\mathcal{R}$ denotes the Borel σ-algebra on $\mathbb{R}$. In this exercise we will show that:
Claim: Y is $\sigma(X)$-$\mathcal{R}$-measurable $\iff$ there exists an $\mathcal{R}$-$\mathcal{R}$-measurable function $f : \mathbb{R} \to \mathbb{R}$, such that $Y = f \circ X$.

Hint: For (b)–(e), cf. the proof of (1.2.16) in the lecture notes.

(a) Show the $\Longleftarrow$ direction.

(b) Show the $\Longrightarrow$ direction for any Y of the form $Y = 1_A$, where $A \in \sigma(X)$.

(c) Show the $\Longrightarrow$ direction for any Y that is a linear combination of indicator functions, i.e. for Y of the form $Y = \sum_{i=1}^n c_i 1_{A_i}$, where $n \in \mathbb{N}$, $c_1, \dots, c_n \in \mathbb{R}$ and $A_1, \dots, A_n \in \sigma(X)$.

(d) Show the $\Longrightarrow$ direction for any Y such that $Y \geq 0$.

(e) Complete the proof of the claim (i.e. show the $\Longrightarrow$ direction for an arbitrary Y).

Submission: until 12:00, Sep 29., through the webpage of the course. You should carefully follow the **submission instructions** on the webpage to get your solutions back.

Office hours: See the webpage for detailed information

- Präsenz (Group 3): Mon. and Thu., 12:00-13:00 in HG G32.6. with previous reservation.
- Probability Theory Assistants: Tue. 15:30-16:30 and Wed. 11:00-12:00 via Zoom with a 10 minutes slot reservation.

Exercise class: Online. In-person exercise classes need previous registration each week.

Exercise sheets and further information are also available on:
https://metaphor.ethz.ch/x/2020/hs/401-3601-00L/