Coordinator
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Probability Theory

Exercise Sheet 2

Exercise 2.1 Take $\Omega = \{a, b, c, d\}$, $\mathcal{A} = \mathcal{P}(\Omega)$ and $\mathcal{C} = \big\{\{a, b\}, \{c, d\}, \{a, c\}, \{b, d\}\big\}$. Consider P the equiprobability on Ω and Q the probability measure $\frac{1}{2}(\delta_a + \delta_d)$ (with δ_a the point measure at a, and δ_d the point measure at d).

(a) Show that $\sigma(\mathcal{C}) = \mathcal{A}$, and P and Q agree on $\mathcal{C}$.

(b) Show that $\{A \in \mathcal{A};\ P(A) = Q(A)\}$ is not a σ-algebra.

(c) Is $\mathcal{C}$ a π-system?

Exercise 2.2 Let $(\Omega, \mathcal{A}, P)$ be a probability space and $(A_n)_{n\in\mathbb{N}}$ a sequence of sets from $\mathcal{A}$. We define

$$\bar{A} := \limsup_{n\to\infty} A_n := \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k \quad , \quad \underline{A} := \liminf_{n\to\infty} A_n := \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} A_k.$$

Let 1_B denote the indicator function of $B \in \mathcal{A}$.

(a) Show that $1_{\bar{A}} = \limsup_{n\to\infty} 1_{A_n}$ and that $1_{\underline{A}} = \liminf_{n\to\infty} 1_{A_n}$.

(b) Show that $P[\underline{A}] \leq \liminf_{n\to\infty} P[A_n]$ and that $P[\bar{A}] \geq \limsup_{n\to\infty} P[A_n]$.

Hint: Use a lemma from Section 1.2 in the lecture notes.

Exercise 2.3 In this exercise, we will construct a countably infinite number of independent random variables, without using a product space with an infinite number of factors.

Consider $\Omega = [0, 1)$, equipped with the Borel σ-algebra and the Lebesgue measure P restricted to $[0, 1)$. We define the random variables

$$Y_n : \Omega \to \mathbb{R}\,, \quad n \geq 1\,,$$

by

$$Y_n(\omega) := \begin{cases} 0 & \text{if } \lfloor 2^n\omega \rfloor \text{ is even,} \\ 1 & \text{if } \lfloor 2^n\omega \rfloor \text{ is odd,} \end{cases}$$

where $\lfloor x \rfloor = \max\{z \in \mathbb{Z} \mid z \leq x\}$ denotes the integer part of x.

(a) Use the binary expansion of ω to show that $\omega = \sum_{j\geq 1} Y_j(\omega) 2^{-j}$.

(b) Show that for every $n \geq 1$, Y_n is in fact a random variable.

(c) Show that Y_n , $n \geq 1$, are independent and satisfy $P[Y_n = 0] = P[Y_n = 1] = \frac{1}{2}$.

Hint: You may use the following observation, without proving it:

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $Y_1, Y_2, \ldots$ be random variables on this space, each taking values only in a countable set (that is, for each i there is a countable set S_i such that $P[Y_i \in S_i] = 1$). Assume that

$$P[Y_1 = z_1, Y_2 = z_2, \ldots, Y_n = z_n] = \prod_{i=1}^{n} P[Y_i = z_i] \text{ for all } z_1, \ldots, z_n \in \mathbb{R} \tag{1}$$

holds for all $n \geq 1$. Then, the infinite sequence of random variables $(Y_i)_{i \geq 1}$ is independent.

Exercise 2.4 (Optional.) A non-empty family $\mathcal{C}$ of subsets of a non-empty set Ω is called a λ-*system*, if

(i) $\Omega \in \mathcal{C}$,

(ii) $A, B \in \mathcal{C} : \ B \subset A \Rightarrow A \setminus B \in \mathcal{C}$,

(iii) $A_n \in \mathcal{C}, \ A_n \subset A_{n+1} \Rightarrow \bigcup_n A_n \in \mathcal{C}$.

Show that the definitions of a Dynkin system and a λ-system are equivalent.

Submission: until 12:00, Oct 6., through the webpage of the course. You should carefully follow the **submission instructions** on the webpage to get your solutions back.

Office hours: See the webpage for detailed information

- Präsenz (Group 3): Mon. and Thu., 12:00-13:00 in HG G32.6. with previous reservation.
- Probability Theory Assistants: Tue. 15:30-16:30 and Wed. 11:00-12:00 via Zoom with a 10 minutes slot reservation.

Exercise class: Online. In-person exercise classes need previous registration each week.

Exercise sheets and further information are also available on:
https://metaphor.ethz.ch/x/2020/hs/401-3601-00L/