ETH Zurich HS 2020
D-MATH, D-PHYS
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Coordinator
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Probability Theory

Exercise Sheet 3

Exercise 3.1 Assume that $X_k = \frac{1}{k^2} + \frac{Z_k}{k^{\frac{1}{4}}}$, for $k \geq 1$, where Z_k are i.i.d random variables with $P[Z_k = 1] = P[Z_k = -1] = \frac{1}{4}$ and $P[Z_k = 0] = \frac{1}{2}$. Discuss the convergence of the random series $\sum_{k\geq 1} X_k$.

Exercise 3.2 Let $\mathcal{M}$ be the set of the real-valued random variables on the probability space $(\Omega, \mathcal{A}, P)$. We define on $\mathcal{M}$ an equivalence relation as follows:

$$X \sim Y \quad :\Longleftrightarrow \quad P(X = Y) = 1$$

We denote by $\mathcal{M}/\sim$ the set of equivalence classes in $\mathcal{M}$ with respect to $\sim$ and we denote by $[X]$ the equivalence class of $X \in \mathcal{M}$.

(a) Show that

$$\begin{aligned} d : (\mathcal{M}/\sim) \times (\mathcal{M}/\sim) &\to \mathbb{R} \\ ([X],[Y]) &\mapsto E\left[|X-Y| \wedge 1\right] \end{aligned}$$

is a metric on $\mathcal{M}/\sim$.

(b) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}$ and let X be an element of $\mathcal{M}$. Show that $([X_n])_{n\in\mathbb{N}}$ converges to $[X]$ with respect to the metric d if and only if $(X_n)_{n\in\mathbb{N}}$ converges to X in probability.

Exercise 3.3 Let X_i, $i \geq 1$, be identically distributed, integrable random variables and define $S_n = \sum_{i=1}^n X_i$ for each $n \geq 1$. Show that:

$$\lim_{M\to\infty} \sup_{n\geq 1} E\left[\frac{|S_n|}{n} 1_{\left\{\frac{|S_n|}{n} > M\right\}}\right] = 0.$$

Note: This family $\left\{\frac{|S_n|}{n}, n \geq 1\right\}$ is thus so-called "uniformly integrable". See (3.6.14) in the lecture notes. Thanks to Theorem 3.41 and the strong law of large numbers, one has that: if $X_i, i \geq 1$, are also pairwise independent, (in addition to being identically distributed as in the question), then $\frac{S_n}{n}$ converges P-a.s. and in L^1 towards $E[X_1]$ for $n \to \infty$.

Submission: until 12:00, Oct 13., through the webpage of the course. You should carefully follow the **submission instructions** on the webpage to get your solutions back.

Office hours: See the webpage for detailed information

- Präsenz (Group 3): Mon. and Thu., 12:00-13:00 in HG G32.6. with previous reservation.
- Probability Theory Assistants: Tue. 15:30-16:30 and Wed. 11:00-12:00 via Zoom with a 10 minutes slot reservation.

Exercise class: Online. In-person exercise classes need previous registration each week.

Exercise sheets and further information are also available on:
https://metaphor.ethz.ch/x/2020/hs/401-3601-00L/