Probability Theory

Exercise Sheet 4

Exercise 4.1 (A version of the Glivenko-Cantelli Theorem)
Let $(X_i)_{i\geq 1}$ be real-valued, i.i.d. random variables on $(\Omega, \mathcal{A}, P)$ with continous distribution function $F : \bar{\mathbb{R}} \to [0,1]$. We define the empirical distribution by

$$\begin{aligned} F_n : \bar{\mathbb{R}} &\to [0,1] \\ x &\mapsto \frac{1}{n}\sum_{i=1}^{n} 1_{\{X_i \leq x\}}. \end{aligned}$$

Show that,

$$\sup_{x\in\mathbb{R}} |F_n(x) - F(x)| \overset{n\to\infty}{\longrightarrow} 0 \quad P\text{ -a.s.}$$

Hint: Show as an intermediate step that for every continuous and non-decreasing function $F : \bar{\mathbb{R}} \to [0,1]$ and every sequence $(F_n : \bar{\mathbb{R}} \to [0,1])_{n\geq 1}$ of non-decreasing functions it holds that if $F_n(x) \overset{n\to\infty}{\longrightarrow} F(x)$ for all $x \in \bar{\mathbb{Q}} = \mathbb{Q} \cup \{\pm\infty\}$, then $(F_n)_{n\geq 1}$ converge uniformly to F.
Remark: The statement of Glivenko-Cantelli also holds for non-continous distribution functions as well.

Exercise 4.2
Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables in a probability space $(\Omega, \mathcal{A}, P)$. Define the two sequences of random variables $(Y_n)_{n\geq 1}$ and $(M_n)_{n\geq 1}$ by

$$Y_n := \min_{1\leq i\leq n} X_i \quad \text{and} \quad M_n := \max_{1\leq i\leq n} X_i$$

1. Let X_1 be uniformly distributed on the interval $[0,1]$. Show that nY_n converges in distribution to an exponential random variable Z with parameter 1, i.e., the density of Z is $e^{-x}1_{[0,\infty)}(x), x \in \mathbb{R}$.
2. Let X_1 be exponentially distributed with parameter 1. Show that $M_n - \log n$ converges in distribution to a random variable Z with Gumbel distribution, i.e. the density of Z is $e^{-x}\exp(-e^{-x}), x \in \mathbb{R}$.

Exercise 4.3

(a) Let f be a (not necessarily Borel-measurable) function from $\mathbb{R}$ to $\mathbb{R}$. Show that the set of discontinuities of f, defined as

$$U_f := \left\{x \in \mathbb{R} \mid f \text{ is discontinuous in } x\right\},$$

is Borel-measurable.

(b) Assume that $X_n \to X$ in distribution. Let f be measurable and bounded, such that $P[X \in U_f] = 0$. Use (2.2.13) – (2.2.14) from the lecture notes to show that we have

$$E[f(X_n)] \underset{n\to\infty}{\to} E[f(X)].$$

(c) Let f be measurable and bounded on $[0,1]$, with U_f of Lebesgue measure 0. Show that the corresponding Riemann sums converge to the integral of f, i.e.

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) \underset{n\to\infty}{\to} \int_0^1 f(x)\mathrm{d}x.$$

Submission: until 12:00, Oct 20., through the webpage of the course. You should carefully follow the **submission instructions** on the webpage to get your solutions back.

Office hours: See the webpage for detailed information

- Präsenz (Group 3): Mon. and Thu., 12:00-13:00 in HG G32.6. with previous reservation.
- Probability Theory Assistants: Tue. 15:30-16:30 and Wed. 11:00-12:00 via Zoom with a 10 minutes slot reservation.

Exercise class: Online. In-person exercise classes need previous registration each week.

Exercise sheets and further information are also available on:
https://metaphor.ethz.ch/x/2020/hs/401-3601-00L/