

Probability Theory

Exercise Sheet 5

Exercise 5.1

- (a) Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of real random variables converging in probability to a random variable X . Show that $(X_n)_{n \in \mathbb{N}}$ converges to X in distribution.
- (b) The converse does not hold in general. Instead, show that if the sequence $(X_n)_{n \in \mathbb{N}}$ converges in distribution to a *constant* random variable $X = c$, then $(X_n)_{n \in \mathbb{N}}$ converges in probability to c .

Exercise 5.2 Compute the characteristic functions of the following distributions:

- (a) The triangular distribution $(1 - |x|)1_{[-1,1]}(x)dx$.
- (b) The Cauchy distribution $\frac{\alpha}{\pi} \frac{1}{x^2 + \alpha^2} dx$ with parameter $\alpha > 0$.
Hint: Use a contour integral.

Exercise 5.3

- (a) Let μ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, with characteristic function ϕ , and let a be in \mathbb{R} . Show that we have

$$\frac{1}{2T} \int_{-T}^T e^{-ita} \phi(t) dt \xrightarrow{T \rightarrow \infty} \mu(\{a\}).$$

Hint: Use Fubini's theorem, and the Dominated Convergence Theorem.

- (b) Let X, Y be independent random variables, with distribution ν , and characteristic function ψ . Show that we have

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |\psi(t)|^2 dt = P(X = Y) = \sum_{x \in \mathbb{R}} \nu(\{x\})^2,$$

where we sum over all $x \in \mathbb{R}$, with $\nu(\{x\}) > 0$, on the right-hand side. (Prove that there can be at most countably many such x .)

Hint: Apply part (a) to the distribution of $X - Y$.

Submission: until 12:00, Oct 27., through the webpage of the course. You should carefully follow the **submission instructions** on the webpage to get your solutions back.

Office hours: See the webpage for detailed information

- Präsenz (Group 3): Mon. and Thu., 12:00-13:00 in HG G32.6. with previous reservation.

- Probability Theory Assistants: Tue. 15:30-16:30 and Wed. 11:00-12:00 via Zoom with a 10 minutes slot reservation.

Exercise class: Online. In-person exercise classes need previous registration each week.

Exercise sheets and further information are also available on:
<https://metaphor.ethz.ch/x/2020/hs/401-3601-00L/>