ETH Zurich HS 2020
D-MATH, D-PHYS
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Coordinator
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Probability Theory

Exercise Sheet 7

Exercise 7.1 Let X and Y be two independent Bernoulli distributed random variables with parameter p. Define $Z = 1_{\{X+Y=0\}}$ and $\mathcal{G} = \sigma(Z)$. Find $E[X|\mathcal{G}]$ and $E[Y|\mathcal{G}]$. Are these random variables also independent?

Exercise 7.2 Let Y and Z be independent random variables on $(\Omega, \mathcal{A}, P)$ with respective distributions μ and ν, and $f : \mathbb{R}^2 \to \mathbb{R}$ be a bounded measurable function. Let $X = f(Y, Z)$, and $h : \mathbb{R} \to \mathbb{R}$ be the bounded measurable function

$$h(y) = \int_{\mathbb{R}} f(y, z) d\nu(z), \text{ for } y \in \mathbb{R}.$$

Show that $E[X|\sigma(Y)] = h(Y)$ P-a.s.

Exercise 7.3 Let S be a random variable with $P[S > t] = e^{-t}$, for all $t > 0$. Calculate the conditional expectation $E\big[S \,\big|\, S \wedge t\big]$, where $S \wedge t := \min(S, t)$ for arbitrary $t > 0$.

Remark: Recall that by definition $E[X|Y] := E[X|\sigma(Y)]$ when X and Y are random variables in the same probability space and X is integrable.

Exercise 7.4 (Optional.) In this exercise we prove that in Theorem 1.37 (Kolmogorov's Three Series Theorem) $(1.4.16) \Rightarrow (1.4.17)$.

Consider X_k, $k \geq 1$ independent random variables and $A > 0$. Set $Y_k := X_k 1_{\{|X_k| \leq A\}}$, $k \geq 1$. Assume that $\sum_k X_k$ converges P-a.s.

(a) Show that $P[\liminf_k \{X_k = Y_k\}] = 1$.

(b) Deduce from (a) that $\sum_k P[|X_k| > A] < \infty$ and $\sum_k Y_k$ converges P-a.s.

(c) Show that $\sum_k \mathrm{Var}(Y_k) < \infty$. (**Hint:** use Exercise 6.3.)

(d) Show that $\sum_k E[Y_k]$ converges. (**Hint:** use Theorem 1.34, moreover (c) and (b).)

Submission: until 12:00, Nov. 10, through the webpage of the course. You should carefully follow the **submission instructions** on the webpage to get your solutions back.

Office hours: See the webpage for detailed information

- Präsenz (Group 3): Mon. and Thu., 12:00-13:00 in HG G32.6. with previous reservation.
- Probability Theory Assistants: Tue. 15:30-16:30 and Wed. 11:00-12:00 via Zoom with a 10 minutes slot reservation.

Exercise class: Online. In-person exercise classes need previous registration each week.

Exercise sheets and further information are also available on:
https://metaphor.ethz.ch/x/2020/hs/401-3601-00L/