Coordinator
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Probability Theory

Exercise Sheet 8

Exercise 8.1 Let X be a random variable in $L^2(\Omega, \mathcal{A}, P)$ and $\mathcal{F} \subseteq \mathcal{A}$. The *conditional variance* of X given $\mathcal{F}$ is defined as $\mathrm{Var}[X|\mathcal{F}] := E[(X - E[X|\mathcal{F}])^2|\mathcal{F}]$. Prove that

(a) $\mathrm{Var}[X|\mathcal{F}] = E[X^2|\mathcal{F}] - E[X|\mathcal{F}]^2$;

(b) $\mathrm{Var}(X) = E[\mathrm{Var}[X|\mathcal{F}]] + \mathrm{Var}[E[X|\mathcal{F}]]$.

(c) Compute $\mathrm{Var}[X|\mathcal{F}]$, where $\mathcal{F} = \sigma(A_1, A_2)$ where $\{A_1, A_2\}$ is a partition of Ω and $P(A_i) > 0$ for $i = 1, 2$.

Exercise 8.2 Let $n \geq 2$, and let $X_1, \ldots, X_n$ be i.i.d. random variables defined on a probability space $(\Omega, \mathcal{A}, P)$.

(a) Show that for every Borel function $g : \mathbb{R}^n \to \mathbb{R}$ with $E[|g(X_1, \ldots, X_n)|] < \infty$ and any permutation π of $\{1, \ldots, n\}$,

$$E[g(X_1, \ldots, X_n)] = E[g(X_{\pi(1)}, \ldots, X_{\pi(n)})].$$

(b) Set $S := X_1 + \ldots + X_n$ and assume that X_1 is integrable. Find a representation of $E[X_1|S]$ as a function of S.

Hint: First show that $E[X_1|S] = E[X_2|S]$ P-a.s.

Exercise 8.3 (Polya's Urn)
An urn initially contains s black and w white balls. We consider the following process. At each step a random ball is drawn from the urn, and is replaced by t balls of the same colour, for some fixed $t \geq 1$. We define the random variable Y_n as the proportion of black balls in the urn after the n-th iteration. Show that $E[Y_{n+1}|\sigma(Y_0, Y_1, \ldots Y_n)] = Y_n$, for all $n \geq 0$, that is, $\{Y_n\}_{n \geq 0}$ is a martingale.

Submission: until 12:00, Nov. 17, through the webpage of the course. You should carefully follow the **submission instructions** on the webpage to get your solutions back.

Office hours: Tue. 15:30-16:30 and Wed. 11:00-12:00 via Zoom with a 10 minutes slot reservation. Organized by the Probability Theory assistants.

Exercise class: Online. Details can be found on the polybox folder of the course.

Exercise sheets and further information are also available on:
https://metaphor.ethz.ch/x/2020/hs/401-3601-00L/