Coordinator
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Probability Theory

Exercise Sheet 9

Exercise 9.1 Let $S, T : \Omega \to \mathbb{N} \cup \{\infty\}$ be $\mathcal{F}_n$-stopping times. Prove or provide a counter example disproving the following statements:

(a) $S - 1$ is a stopping time.

(b) $S + 1$ is a stopping time.

(c) $S \wedge T$ is a stopping time.

(d) $S \vee T$ is a stopping time.

(e) $S + T$ is a stopping time.

Exercise 9.2 Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $(\mathcal{F}_n)_{n\geq 0}$. Let $S \leq T$ be two bounded $(\mathcal{F}_n)_{n\geq 0}$-stopping times and let $(X_n)_{n\geq 0}$ be an $(\mathcal{F}_n)_{n\geq 0}$-submartingale. Show that

$$E[X_T|\mathcal{F}_S] \geq X_S,\ P\text{-a.s.}.$$

(See (3.3.6) on p. 89 of the lecture notes for the definition of $\mathcal{F}_S$.)

Exercise 9.3 Let Y_n, $n \geq 0$ be i.i.d. with $P[Y_0 = 1] = p$ and $P[Y_0 = 0] = 1 - p$ for some $p \in (0, 1)$. Let $\mathcal{F}_n := \sigma(Y_0, \ldots, Y_n)$ for $n \geq 0$ and define

$$T := \inf\{n \geq 0 \mid Y_n = 1\}.$$

Determine the Doob decomposition of $X_n := 1_{\{T\leq n\}}$, $n \geq 0$.
Hint: First check that X_n is an $\mathcal{F}_n$-submartingale. Then try to use Proposition 3.19.

Submission: until 12:00, Nov. 24, through the webpage of the course. You should carefully follow the **submission instructions** on the webpage to get your solutions back.

Office hours: Tue. 15:30-16:30 and Wed. 11:00-12:00 via Zoom with a 10 minutes slot reservation. Organized by the Probability Theory assistants.

Exercise class: Online. Details can be found on the polybox folder of the course.

Exercise sheets and further information are also available on:
https://metaphor.ethz.ch/x/2020/hs/401-3601-00L/