

Probability Theory

Exercise Sheet 10

Exercise 10.1 Let $(X_n)_{n \geq 0}$ be a supermartingale with respect to the natural filtration $(\mathcal{F}_n)_{n \geq 0}$. Suppose $X_n \geq 0$ for all $n \geq 0$ and consider the \mathcal{F}_n -stopping time $T := \inf\{n \geq 0 : X_n = 0\}$. Show that $X_n = 0$ on the event $\{T < n\}$.

Exercise 10.2 Let $(X_i)_{i \geq 1}$ be i.i.d. random variables with mean 0 and variance $\sigma^2 < \infty$, defined in the probability space (Ω, \mathcal{A}, P) . Set $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ and let T be a \mathcal{F}_n -stopping time with $E[T] < \infty$. Define $S_n := \sum_{i=1}^n X_i$ for $n \geq 1$ and $S_0 = 0$.

(a) Show, with the help of the Optional Stopping Theorem, that for all $n \geq 0$,

$$E[S_{T \wedge n}^2] = \sigma^2 E[T \wedge n].$$

(b) Prove that $(S_{T \wedge n})_{n \geq 1}$ is a Cauchy sequence in $L^2(\Omega, \mathcal{A}, P)$.

(c) Show that,

$$E[S_T^2] = \sigma^2 E[T].$$

Exercise 10.3 Consider a probability space (Ω, \mathcal{F}, P) equipped with a filtration $\{\mathcal{F}_n\}_{n \geq 0}$, and let X_n be an \mathcal{F}_n -martingale for which $|X_{n+1} - X_n| \leq M$ P -a.s. for some fixed $M < \infty$. Define the events C, D by

$$C := \{\lim X_n \text{ exists and is finite}\},$$
$$D := \{\limsup X_n = +\infty \text{ and } \liminf X_n = -\infty\}.$$

Show that $P[C \cup D] = 1$.

Hint: Show that $P[C^c \cap (\{\sup_{n \in \mathbb{N}} X_n < a\} \cup \{\inf_{n \in \mathbb{N}} X_n > -a\})] = 0$, for all $a > 0$, by considering the processes $\{X_{T_A \wedge n}\}_{n \geq 0}$, for $A = [a, \infty)$ and $A = (-\infty, -a]$, where $T_A = \inf\{n \geq 0 : X_n \in A\}$.

Submission: until 12:00, Dec. 1, through the webpage of the course. You should carefully follow the **submission instructions** on the webpage to get your solutions back.

Office hours: Tue. 15:30-16:30 and Wed. 11:00-12:00 via Zoom with a 10 minutes slot reservation. Organized by the Probability Theory assistants.

Exercise class: Online. Details can be found on the polybox folder of the course.

Exercise sheets and further information are also available on:
<https://metaphor.ethz.ch/x/2020/hs/401-3601-00L/>