Coordinator
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Probability Theory

Exercise Sheet 11

Exercise 11.1 (The generalized Borel-Cantelli lemma)
Consider $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}_n\}_{n\geq 0}$, and let $A_n \in \mathcal{F}_n$, $n \geq 1$, be a sequence of events. Show that, up to a P-nullset,

$$\limsup_{n\to\infty} A_n = \{\sum_{n\geq 1} P[A_n|\mathcal{F}_{n-1}] = \infty\}.$$

Hint: Use Exercise 10.3.

Exercise 11.2 Consider $Y, X_i, i \geq 1$, independent random variables with $Y \geq 0$, integer valued such that $E[Y] = \mu \in (1, \infty)$, and $X_i, i \geq 1$, i.i.d. Bernoulli random variables with $P[X_i = 0] = q \in (0, 1)$. If $S_m, m \geq 0$, denotes the partial sums of the X_i, let ν be the law of S_Y. Consider the Galton-Watson chain $Z_n, n \geq 0$ with offspring distribution ν (see p. 97 of the Lecture Notes).

(a) For which values of q is the Galton-Watson chain subcritical?

(b) If Y is constant and equal to 2, find

$$f(q) := P[Z_n > 0, \text{ for all } n \geq 0].$$

Hint: See Lecture Notes p. 100.

Exercise 11.3 Let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of independent, non-negative random variables with expectation 1. Consider the natural filtration $(\mathcal{F}_n)_{n\geq 0}$. We define

$$M_0 = 1, \quad M_n = Y_1 Y_2 \cdots Y_n, \text{ for } n \in \mathbb{N}.$$

(a) Prove that $(M_n)_{n\in\mathbb{N}}$ is a non-negative martingale with respect to the filtration $(\mathcal{F}_n)_{n\geq 0}$ and there exists a random variable M_∞, so that $M_n \to M_\infty$ a.s.

(b) Let $a_n := E[\sqrt{Y_n}]$. Show that $a_n \in (0, 1]$.

(c) Show that if $\prod_n a_n > 0$, it holds that $M_n \to M_\infty$ in L^1 and $E[M_\infty] = 1$.
Hint: Let $\hat{Y}_n := \sqrt{Y_n}/a_n$ and $\hat{M}_n := \hat{Y}_1 \hat{Y}_2, \ldots, \hat{Y}_n$ for $n \geq 1$, $\hat{M}_0 = 1$. Note that $M_n \leq \hat{M}_n^2$ for $n \in \mathbb{N}$. Then use (a) together with Doob's inequality to conclude the proof.

(d) Show that if $\prod_n a_n = 0$, then $M_\infty = 0$ a.s.

Submission: until 12:00, Dec. 8, through the webpage of the course. You should carefully follow the **submission instructions** on the webpage to get your solutions back.

Office hours: Tue. 15:30-16:30 and Wed. 11:00-12:00 via Zoom with a 10 minutes slot reservation. Organized by the Probability Theory assistants.

Exercise class: Online. Details can be found on the polybox folder of the course.

Exercise sheets and further information are also available on:
https://metaphor.ethz.ch/x/2020/hs/401-3601-00L/