ETH Zurich HS 2020
D-MATH, D-PHYS
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Coordinator
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Probability Theory

Exercise Sheet 12

Exercise 12.1 Let X_n, $n \geq 0$, be a uniformly integrable submartingale and N a stopping time.

(a) Show that $\sup_n E[X_{N\wedge n}^+] \leq \sup_n E[X_n^+] < \infty$.

(b) Show that X_N (where $X_N 1_{\{N=\infty\}} = 1_{\{N=\infty\}} \lim_n X_n$) is integrable.

(c) Show that $X_{N\wedge n}$, $n \geq 0$, is a uniformly integrable submartingale.

(d) Show that $X_{N\wedge n}$ converges P-a.s. and in L^1 to X_N.

Exercise 12.2 Let $(X_n)_{n\geq 0}$ be a uniformly integrable family of random variables on $(\Omega, \mathcal{A}, P)$.

(a) Assume that X_n converges to a random variable X in distribution. Show that

$$E[X_n] \xrightarrow{n\to\infty} E[X].$$

Remark: Compare to (3.6.18)–(3.6.20), p. 112 of the lecture notes.

(b) Assume that X_n converges to a random variable X in probability. Show that $X \in L^1$ and that X_n converges to X in L^1.

Exercise 12.3 *Azuma's inequality.* Let $0 = X_0, \dots, X_m$ be a martingale with $|X_{i+1} - X_i| \leq 1$ for all $0 \leq i < m$. Let $\lambda > 0$ be arbitrary.

(a) Show that $E[e^{\alpha(X_i - X_{i-1})} | X_{i-1}, X_{i-2}, \dots, X_0] \overset{(1)}{\leq} \cosh\alpha \overset{(2)}{\leq} e^{\alpha^2/2}$.

Hint: For (1) use that for $y \in [-1,1]$, $e^{\lambda y} \leq \frac{e^\lambda + e^{-\lambda}}{2} + y\frac{e^\lambda - e^{-\lambda}}{2}$. Inequality (2) follows from the series expansion of $\cosh\alpha$.

(b) Show that $E[e^{\alpha X_m}] \leq e^{\alpha^2 m/2}$.

(c) Show that $P\left[X_m > \lambda\sqrt{m}\right] < e^{-\lambda^2/2}$.

Submission: until 12:00, Dec. 15, through the webpage of the course. You should carefully follow the **submission instructions** on the webpage to get your solutions back.

Office hours: Tue. 15:30-16:30 and Wed. 11:00-12:00 via Zoom with a 10 minutes slot reservation. Organized by the Probability Theory assistants.

Exercise class: Online. Details can be found on the polybox folder of the course.

Exercise sheets and further information are also available on:
https://metaphor.ethz.ch/x/2020/hs/401-3601-00L/