ETH Zurich HS 2020
D-MATH, D-PHYS
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Probability Theory

Exercise Sheet 13

Definition: Let $(\Omega, \mathcal{F}, (P_x)_{x\in E})$ be a canonical (time-homogenous) Markov chain with a *countable* state space E, a transition kernel K, and canonical coordinates $(X_n)_{n\geq 0}$. The matrix

$$Q = (Q(x,y))_{x,y\in E} := (K(x,\{y\}))_{x,y\in E} = (P_x[X_1 = y])_{x,y\in E}$$

is then called the *transition matrix* of the Markov chain. For the meanings of notation P_x and transition kernel we refer to p. 145 in lecture notes.

Exercise 13.1 Let $(\Omega, \mathcal{F}, (P_x)_{x\in E})$ be a canonical time–homogeneous Markov chain with a countable state space E, canonical coordinate process $(X_n)_{n\geq 0}$ and transition kernel K. Let $A \subset E$ and τ_A the first entrance time of A, i.e., $\tau_A := \inf\{n \geq 0 \,|\, X_n \in A\}$. Suppose that there exists $n \geq 1$ and $\alpha > 0$ such that for all $x \in A^c$,

$$P_x[X_n \in A] = \sum_{a\in A} P_x[X_n = a] \geq \alpha.$$

Show that for all $x \in E$ we have that $P_x(\tau_A < +\infty) = 1$.

Exercise 13.2 Let E be a countable set, $(S, \mathcal{S})$ a measurable space, $(Y_n)_{n\geq 1}$ a sequence of i.i.d. S-valued random variables. We define a sequence $(Z_n)_{n\geq 0}$ through $Z_0 = x \in E$ and $Z_{n+1} = \Phi(Z_n, Y_{n+1})$, where $\Phi : E \times S \to E$ is a measurable map. Find a transition kernel K on E such that the canonical law P_x with transition kernel K has the same law as $(Z_n)_{n\geq 0}$ (hence $(Z_n)_{n\geq 0}$ induces a time-homogenous Markov chain with transition kernel K). Calculate the corresponding transition matrix.

Exercise 13.3 *(Probabilistic solution to the Dirichlet problem).*
Consider $(X_n)_{n\geq 0}$ the canonical Markov chain on $\mathbb{Z}^d$ with transition kernel

$$K(x, dy) = \frac{1}{2d} \sum_{e\in\mathbb{Z}^d : |e|=1} \delta_{x+e}(dy),$$

corresponding to the simple random walk on $\mathbb{Z}^d$. Let $U \neq \emptyset$ be a finite subset of $\mathbb{Z}^d$.

(a) If $T_U = \inf\{n \geq 0; X_n \notin U\}$ stands for the exit time of U, show that for all $x \in \mathbb{Z}^d$, P_x-a.s., $T_U < \infty$.

Hint: Show that $M_n = \sum_{1\leq i\leq d} X_n \cdot e_i$, $n \geq 0$ (with $e_1, \dots, e_d$ the canonical basis of $\mathbb{Z}^d$) is a martingale with bounded increments and use Exercise 10.3.

(b) Let g be a bounded function on $\mathbb{Z}^d \setminus U$. If $f : \mathbb{Z}^d \to \mathbb{R}$ solves the Dirichlet problem

$$(*) \begin{cases} \frac{1}{2d} \sum_{y:|y-x|=1} f(y) = f(x), & \text{for } x \in U, \\ f(x) = g(x), & \text{for } x \notin U. \end{cases}$$

Show that necessarily $f(x) = E_x[g(X_{T_U})]$ for all $x \in \mathbb{Z}^d$.

Hint: Use the martingale (4.2.58) in the lecture notes and the Optional Stopping Theorem.

(c) Show, without using (b), that the function $f(x) = E_x[g(X_{T_U})]$, $x \in \mathbb{Z}^d$ solves $(*)$.

Hint: distinguish the cases $x \notin U$ and $x \in U$. When $x \in U$ note that P_x-a.s., $g(X_{T_U}) = g(X_{T_U}) \circ \theta_1$ and use the Markov property (4.2.55).

Submission: until 12:00, Dec. 22, through the webpage of the course. You should carefully follow the **submission instructions** on the webpage to get your solutions back.

Office hours: Tue. 15:30-16:30 and Wed. 11:00-12:00 via Zoom with a 10 minutes slot reservation. Organized by the Probability Theory assistants.

Exercise class: Online. Details can be found on the polybox folder of the course.

Exercise sheets and further information are also available on:
https://metaphor.ethz.ch/x/2020/hs/401-3601-00L/