Probability Theory

Exercise Sheet 2

Exercise 2.1 Take $\Omega = \{a, b, c, d\}$, $\mathcal{A} = \mathcal{P}(\Omega)$ and $\mathcal{C} = \big\{\{a, b\}, \{c, d\}, \{a, c\}, \{b, d\}\big\}$. Consider P the equiprobability on Ω and Q the probability measure $\frac{1}{2}(\delta_a + \delta_d)$ (with δ_a the point measure at a, and δ_d the point measure at d).

(a) Show that $\sigma(\mathcal{C}) = \mathcal{A}$, and P and Q agree on $\mathcal{C}$.

(b) Show that $\{A \in \mathcal{A};\ P(A) = Q(A)\}$ is not a σ-algebra.

(c) Is $\mathcal{C}$ a π-system?

Exercise 2.2 Let $(\Omega, \mathcal{A}, P)$ be a probability space and $(A_n)_{n\in\mathbb{N}}$ a sequence of sets from $\mathcal{A}$. We define

$$\bar{A} := \limsup_{n\to\infty} A_n := \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k \quad , \quad \underline{A} := \liminf_{n\to\infty} A_n := \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} A_k .$$

Let 1_B denote the indicator function of $B \in \mathcal{A}$.

(a) Show that $1_{\bar{A}} = \limsup_{n\to\infty} 1_{A_n}$ and that $1_{\underline{A}} = \liminf_{n\to\infty} 1_{A_n}$.

(b) Show that $P[\underline{A}] \leq \liminf_{n\to\infty} P[A_n]$ and that $P[\bar{A}] \geq \limsup_{n\to\infty} P[A_n]$.

Hint: Use a lemma from Section 1.2 in the lecture notes.

Exercise 2.3 In this exercise, we will construct a countably infinite number of independent random variables, without using a product space with an infinite number of factors.

Consider $\Omega = [0, 1)$, equipped with the Borel σ-algebra and the Lebesgue measure P restricted to $[0, 1)$. We define the random variables

$$Y_n : \Omega \to \mathbb{R}\,, \quad n \geq 1\,,$$

by

$$Y_n(\omega) := \begin{cases} 0 & \text{if } \lfloor 2^n \omega \rfloor \text{ is even,} \\ 1 & \text{if } \lfloor 2^n \omega \rfloor \text{ is odd,} \end{cases}$$

where $\lfloor x \rfloor = \max\{z \in \mathbb{Z} \mid z \leq x\}$ denotes the integer part of x.

(a) Use the binary expansion of ω to show that $\omega = \sum_{j\geq 1} Y_j(\omega) 2^{-j}$.

(b) Show that for every $n \geq 1$, Y_n is in fact a random variable.

(c) Show that Y_n , $n \geq 1$, are independent and satisfy $P[Y_n = 0] = P[Y_n = 1] = \frac{1}{2}$.

Hint: You may use the following observation, without proving it:

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $Y_1, Y_2, \dots$ be random variables on this space, each taking values only in a countable set (that is, for each i there is a countable set S_i such that $P[Y_i \in S_i] = 1$). Assume that

$$P[Y_1 = z_1, Y_2 = z_2, \dots, Y_n = z_n] = \prod_{i=1}^{n} P[Y_i = z_i] \text{ for all } z_1, \dots, z_n \in \mathbb{R} \tag{1}$$

holds for all $n \geq 1$. Then, the infinite sequence of random variables $(Y_i)_{i \geq 1}$ is independent.

Exercise 2.4 (Optional.) A non-empty family $\mathcal{C}$ of subsets of a non-empty set Ω is called a λ-*system*, if

(i) $\Omega \in \mathcal{C}$,

(ii) $A, B \in \mathcal{C} :\ B \subset A \Rightarrow A \setminus B \in \mathcal{C}$,

(iii) $A_n \in \mathcal{C},\ A_n \subset A_{n+1} \Rightarrow \bigcup_n A_n \in \mathcal{C}$.

Show that the definitions of a Dynkin system and a λ-system are equivalent.

Submission: until 12:00, Oct 6., through the webpage of the course. You should carefully follow the **submission instructions** on the webpage to get your solutions back.

Office hours: See the webpage for detailed information

- Präsenz (Group 3): Mon. and Thu., 12:00-13:00 in HG G32.6. with previous reservation.
- Probability Theory Assistants: Tue. 15:30-16:30 and Wed. 11:00-12:00 via Zoom with a 10 minutes slot reservation.

Exercise class: Online. In-person exercise classes need previous registration each week.

Exercise sheets and further information are also available on:
https://metaphor.ethz.ch/x/2020/hs/401-3601-00L/

Solution 2.1

(a) We start with the first claim. Because $\{a\} = \{a, b\} \cap \{a, c\}$, we know that $\{a\} \in \sigma(\mathcal{C})$. By cyclic symmetry we obtain that $\{b\}, \{c\}, \{d\} \in \sigma(\mathcal{C})$ as well. The first claim follows now from Exercise 1.2 (a). For the second claim, we simply observe that $\forall B \in \mathcal{C}$, $P(B) = Q(B) = 1/2$.

(b) Suppose $\{A \in \mathcal{A};\ P(A) = Q(A)\}$ is a σ-algebra. Since this collection contains $\mathcal{C}$, by (a), it would contain also $\mathcal{A}$, by (a). Thus, P and Q would be equal, which is a contradiction.

(c) No. By a direct inspection, we see that

$$\{a, b\} \cap \{a, c\} = \{a\} \notin \mathcal{C}.$$

We can also show this by the following argument: If it were a π-system, by (1.3.11) in the lecture notes, any P and Q agreeing on $\mathcal{C}$ would be equal.

Solution 2.2

(a) Let $\omega \in \Omega$. We will show

$$\omega \in \bar{A} \Leftrightarrow \limsup_{n\to\infty} 1_{A_n}(\omega) = 1. \tag{2}$$

The case on $\bar{A}^c$ is handled analogously.

Let $\limsup_{n\to\infty} 1_{A_n}(\omega) = 1$. Then for all $n \in \mathbb{N}$ we have

$$1 = \limsup_{m\to\infty} 1_{A_m}(\omega) = \inf_{m\in\mathbb{N}} \sup_{k\geq m} 1_{A_k}(\omega) \leq \sup_{k\geq n} 1_{A_k}(\omega) \leq 1,$$

and thus $\sup_{k\geq n} 1_{A_k}(\omega) = 1$. Since the indicator function 1 only takes the values 0 and 1, there exists for each $n \in \mathbb{N}$ a $k \geq n$ such that $\omega \in A_k$. In other words, $\omega \in \bar{A}$.

If on the other hand $\omega \in \bar{A}$, then there exists for all $n \in \mathbb{N}$ a $k \geq n$ for which $\omega \in A_k$. Thus $\sup_{k\geq n} 1_{A_k}(\omega) = 1$ for every $n \in \mathbb{N}$, implying $\limsup_{n\to\infty} 1_{A_n}(\omega) = 1$.

Thus, we have shown (2), and $1_{\bar{A}} = \limsup_{n\to\infty} 1_{A_n}$ follows. To show the analogous result for the lim inf we note that

$$(\underline{A})^c = \left(\bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} A_k \right)^c = \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k^c = \limsup_{n\to\infty} A_n^c.$$

Thus we can deduce the result for lim inf from the already proven result for lim sup.

$$\begin{aligned} 1_{\underline{A}} &= 1 - 1_{(\underline{A})^c} = 1 - 1_{\limsup_{n\to\infty} A_n^c} \\ &= 1 - \limsup_{n\to\infty} 1_{A_n^c} = \liminf_{n\to\infty} (1 - 1_{A_n^c}) = \liminf_{n\to\infty} 1_{A_n}. \end{aligned}$$

(b) These inequalities are immediate consequences of (a) and Fatou's lemma.

Solution 2.3

(a) Let us write down the binary representation of ω, i.e.

$$\begin{aligned} \omega &= \sum_{j\geq 1} \omega_j 2^{-j}, \quad \omega_j \in \{0, 1\}, \\ &= 0.\omega_1\omega_2 \ldots \end{aligned}$$

Technical point: In cases like $\omega = 1/2$, which can be represented as both $0.1000\ldots$ and $0.01111\ldots$, we choose the terminating binary representation, i.e the one which "ends" in an infinite sequence of zeroes, which is the usual convention.

$$\text{For } \omega = 0.\omega_1\omega_2\ldots\omega_j\ldots \Rightarrow 2^j\omega = \omega_1\omega_2\ldots\omega_j.\omega_{j+1}\ldots \in [\omega_1\omega_2\ldots\omega_j,\ \omega_1\omega_2\ldots\omega_j+1),$$
$$\Rightarrow \lfloor 2^j\omega\rfloor = \omega_1\ldots\omega_j \Rightarrow \begin{cases} \lfloor 2^j\omega\rfloor = \omega_1\ldots\omega_j \text{ is odd}, & \Rightarrow \omega_j = 1, \\ \lfloor 2^j\omega\rfloor = \omega_1\ldots\omega_j \text{ is even}, & \Rightarrow \omega_j = 0. \end{cases}$$

Hence we have $Y_j(\omega) = \omega_j$. Since this holds for every $j \geq 1$, we get the desired result.

(b) From the representation in part (a), we see that for $n \geq 1$

$$\{Y_n = 0\} = \Omega \cap \bigcup_{j=0}^{2^{n-1}-1} \left[\frac{2j}{2^n}, \frac{2j+1}{2^n}\right). \tag{3}$$

Similarly, we have that $\{Y_n = 1\}$ belongs to the Borel σ-algebra of $[0,1)$. Thus the Y_n are measurable.

(c) From (3) we have that $P[Y_n = 0] = 2^{n-1}/(2^n) = 1/2 = P[Y_n = 1]$. To prove independence, we note that for $n \geq 1$ and $z_1, z_2, \ldots, z_n \in \{0,1\}$, we have

$$P\left[\bigcap_{j=1}^n \{Y_j = z_j\}\right] = P\left[\left[\sum_{j=1}^n \frac{z_j}{2^j}, \sum_{j=1}^n \frac{z_j}{2^j} + \frac{1}{2^n}\right)\right] = 2^{-n} = \prod_{j=1}^n P[Y_j = z_j].$$

By the observation given in the hint, this implies independence of the infinite sequence $\{Y_n\}$, $n \geq 1$.

Solution 2.4 "$\Leftarrow$" Let $\mathcal{C}$ be a λ-system, then:

- $\Omega \in \mathcal{C}$, because of (i).
- Let A be in $\mathcal{C}$, $A \subset \Omega \overset{\text{(ii)}}{\Rightarrow} A^c = \Omega \setminus A \in \mathcal{C}$.
- Let $A, B \in \mathcal{C}$ disjoint sets. Then we have that $A \subset B^c \in \mathcal{C}$ and due to (ii):

$$B^c \setminus A \in \mathcal{C} \Rightarrow (B^c \setminus A)^c = B \cup A \in \mathcal{C}. \tag{4}$$

Now let $(A_i)_{i\geq 1} \subset \mathcal{C}$ be pairwise disjoint subsets, and set $B_n := \bigcup_{i=1}^n A_i$. By (4), B_n is in $\mathcal{C}$ for every $n \geq 1$, and clearly $B_n \subset B_{n+1}$. Therefore by (iii) we get that,

$$\bigcup_{n\geq 1} B_n = \bigcup_{i\geq 1} A_i \in \mathcal{C}.$$

"$\Rightarrow$" Let $\mathcal{C}$ be a Dynkin-system, then we have:

(i): $\Omega \in \mathcal{C}$.

(ii): Let A, B be in $\mathcal{C}$ with $A \subset B$. Hence $A \cap B^c = \emptyset$, and therefore

$$A \cap B^c = \emptyset \Rightarrow A \cup B^c \in \mathcal{C} \Rightarrow (A \cup B^c)^c = B \setminus A \in \mathcal{C}.$$

(iii): Let $(A_n)_{n\geq 1} \subset \mathcal{C}$ be a sequence satisfying that $A_n \subset A_{n+1}$ for every $n \geq 1$ and set $F_1 = A_1$, $F_n := A_n \setminus A_{n-1} \overset{(ii)}{\in} \mathcal{C}$. Then, $F_n \cap F_k = \emptyset$, for $k \neq n$ and $\bigcup_{n\geq 1} F_n = \bigcup_{k\geq 1} A_k \in \mathcal{C}$.