ETH Zurich HS 2020
D-MATH, D-PHYS
Prof. A.-S. Sznitman

Coordinator
Daniel Contreras

Probability Theory

Exercise Sheet 3

Exercise 3.1 Assume that $X_k = \frac{1}{k^2} + \frac{Z_k}{k^{\frac{1}{4}}}$, for $k \geq 1$, where Z_k are i.i.d random variables with $P[Z_k = 1] = P[Z_k = -1] = \frac{1}{4}$ and $P[Z_k = 0] = \frac{1}{2}$. Discuss the convergence of the random series $\sum_{k \geq 1} X_k$.

Exercise 3.2 Let $\mathcal{M}$ be the set of the real-valued random variables on the probability space $(\Omega, \mathcal{A}, P)$. We define on $\mathcal{M}$ an equivalence relation as follows:

$$X \sim Y \quad :\Longleftrightarrow \quad P(X = Y) = 1$$

We denote by $\mathcal{M}/\!\sim$ the set of equivalence classes in $\mathcal{M}$ with respect to $\sim$ and we denote by $[X]$ the equivalence class of $X \in \mathcal{M}$.

(a) Show that

$$\begin{aligned} d : (\mathcal{M}/\sim) \times (\mathcal{M}/\sim) &\to \mathbb{R} \\ ([X],[Y]) &\mapsto E\left[|X - Y| \wedge 1\right] \end{aligned}$$

is a metric on $\mathcal{M}/\!\sim$.

(b) Let $(X_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{M}$ and let X be an element of $\mathcal{M}$. Show that $([X_n])_{n \in \mathbb{N}}$ converges to $[X]$ with respect to the metric d if and only if $(X_n)_{n \in \mathbb{N}}$ converges to X in probability.

Exercise 3.3 Let X_i, $i \geq 1$, be identically distributed, integrable random variables and define $S_n = \sum_{i=1}^n X_i$ for each $n \geq 1$. Show that:

$$\lim_{M \to \infty} \sup_{n \geq 1} E\left[\frac{|S_n|}{n} 1_{\left\{\frac{|S_n|}{n} > M\right\}}\right] = 0.$$

Note: This family $\left\{\frac{|S_n|}{n}, n \geq 1\right\}$ is thus so-called "uniformly integrable". See (3.6.14) in the lecture notes. Thanks to Theorem 3.41 and the strong law of large numbers, one has that: if $X_i, i \geq 1$, are also pairwise independent, (in addition to being identically distributed as in the question), then $\frac{S_n}{n}$ converges P-a.s. and in L^1 towards $E[X_1]$ for $n \to \infty$.

Submission: until 12:00, Oct 13., through the webpage of the course. You should carefully follow the **submission instructions** on the webpage to get your solutions back.

Office hours: See the webpage for detailed information

- Präsenz (Group 3): Mon. and Thu., 12:00-13:00 in HG G32.6. with previous reservation.
- Probability Theory Assistants: Tue. 15:30-16:30 and Wed. 11:00-12:00 via Zoom with a 10 minutes slot reservation.

Exercise class: Online. In-person exercise classes need previous registration each week.

Exercise sheets and further information are also available on:
https://metaphor.ethz.ch/x/2020/hs/401-3601-00L/

Solution 3.1 Using the same notations as in the statement of Theorem 1.37 (Kolmogorov's three-series theorem), we choose $A = 2$ and define $Y_k := X_k 1_{\{|X_k| \le A\}}$ for $k \ge 1$. From the definition of X_k and the fact that $|Z_k| \le 1$ we actually have $|X_k| \le 2$ and thus $Y_k = X_k$ for all $k \ge 1$.
Since $\mathrm{Var}(Z_k) = \frac{1}{2}$ for all $k \ge 1$, we have $\mathrm{Var}(X_k) = \mathrm{Var}(\frac{1}{k^2} + \frac{Z_k}{k^{\frac{1}{4}}}) = \frac{1}{k^{\frac{1}{2}}}\mathrm{Var}(Z_k) = \frac{1}{2k^{\frac{1}{2}}}$. Hence, it holds that $\sum_{k \ge 1} \mathrm{Var}(Y_k) = \sum_{k \ge 1} \frac{1}{2k^{\frac{1}{2}}} = \infty$, which implies that the condition iii) in (1.4.17) fails. By Theorem 1.37, we then obtain that $\sum_{k \ge 1} X_k$ cannot converge P-a.s., or in other words, $P[\sum_{k \ge 1} X_k \text{ converges}] < 1$. Since the event $\{\sum_{k \ge 1} X_k \text{ converges}\}$ belongs to the asymptotic σ-algebra $\mathcal{F}_\infty$ associated with independent random variables $X_k.k \ge 1$, by Theorem 1.30 (Kolmogorov's 0-1 law) we can conclude that $P[\sum_{k \ge 1} X_k \text{ converges}] = 0$.

Solution 3.2

(a) We verify the criteria for d to be a metric

1. It is clear that d is well-defined;
2. From the definition of d we know that $\forall X, Y \ d([X], [Y]) = d([Y], [X])$;
3. It also follows from the definition of d that $\forall X \ d([X], [X]) = 0$;
4. Observe that $d([X], [Y]) = 0$ for $X, Y \in L^0$ implies $X = Y$ P-.a.s., which further implies $[X] = [Y]$ in $\mathcal{M}/\sim$;
5. To prove that $\forall X, Y, Z \in L^0 \ d([X], [Z]) \le d([X], [Y]) + d([Y], [Z])$, it is sufficient to note that for all $a, b, c \in \mathbb{R}$,

$$|a - c| \wedge 1 \le |a - b| \wedge 1 + |b - c| \wedge 1.$$

(b) Assume $d([X_n], [X]) \to 0$. By Chebyshev's inequality (1.2.13), it follows that

$$P[|X_n - X| > \varepsilon] = P[|X_n - X| \wedge 1 > \varepsilon] \le \frac{E[|X_n - X| \wedge 1]}{\varepsilon} \to 0.$$

Conversely, assume $P[|X_n - X| > \varepsilon] \to 0$ for each $\varepsilon > 0$. Then, it follows that

$$\begin{aligned} E[|X_n - X| \wedge 1] &\le E[|X_n - X| \wedge 1, |X_n - X| < \varepsilon] \\ &\quad + E[|X_n - X| \wedge 1, |X_n - X| \ge \varepsilon] \\ &\le \varepsilon + P[|X_n - X| \ge \varepsilon] < 2\varepsilon, \end{aligned}$$

for sufficiently large n.

Solution 3.3 Let $\widetilde{S}_n = \sum_{i=1}^n |X_i|$. Since $\widetilde{S}_n \ge |S_n|$, we have,

$$1_{\left\{\frac{|S_n|}{n} > M\right\}} \le 1_{\left\{\frac{\widetilde{S}_n}{n} > M\right\}}$$

which implies that

$$E\left[\frac{|S_n|}{n} 1_{\left\{\frac{|S_n|}{n} > M\right\}}\right] \le E\left[\frac{\widetilde{S}_n}{n} 1_{\left\{\frac{\widetilde{S}_n}{n} > M\right\}}\right].$$

Hence we can assume, without loss of generality, that $X_i \geq 0$ for all i. Then we have that for $A > 0$:

$$\begin{aligned}
E\left[\frac{S_n}{n} 1_{\left\{\frac{S_n}{n} > M\right\}}\right] &= E\left[\frac{1}{n}\Big(\sum_{i=1}^n X_i 1_{\{X_i > A\}}\Big) 1_{\left\{\frac{S_n}{n} > M\right\}}\right] + E\left[\frac{1}{n}\Big(\sum_{i=1}^n X_i 1_{\{X_i \leq A\}}\Big) 1_{\left\{\frac{S_n}{n} > M\right\}}\right] \\
&\leq E\left[\frac{1}{n}\sum_{i=1}^n X_i 1_{\{X_i > A\}}\right] + E\left[\frac{1}{n}\sum_{i=1}^n A\, 1_{\left\{\frac{S_n}{n} > M\right\}}\right] \\
&= E\left[X_1 1_{\{X_1 > A\}}\right] + A\ P\left[\frac{S_n}{n} > M\right] \\
&\stackrel{(*)}{\leq} E\left[X_1 1_{\{X_1 > A\}}\right] + \frac{A}{M}\, E\left[\frac{S_n}{n}\right] \\
&= E\left[X_1 1_{\{X_1 > A\}}\right] + \frac{A}{M}\, E[X_1],
\end{aligned}$$

where we have used the fact that $X_i \geq 0$ for all i and applied Chebyshev's inequality (1.2.13) at $(*)$.

Now we take $A = \sqrt{M}$. Then:

$$\varlimsup_{M\to\infty} \sup_{n\geq 1} E\left[\frac{S_n}{n} 1_{\{\frac{S_n}{n} > M\}}\right] \leq \varlimsup_{M\to\infty} E\left[X_1 1_{\{X_1 > \sqrt{M}\}}\right] + \varlimsup_{M\to\infty} \frac{1}{\sqrt{M}}\, E[X_1] = 0.$$

Where the last equality follows by dominated convergence and the fact that X_1 is integrable.