Coordinator
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Probability Theory

Exercise Sheet 4

Exercise 4.1 (A version of the Glivenko-Cantelli Theorem)
Let $(X_i)_{i\geq 1}$ be real-valued, i.i.d. random variables on $(\Omega, \mathcal{A}, P)$ with continous distribution function $F : \bar{\mathbb{R}} \to [0,1]$. We define the empirical distribution by

$$\begin{array}{rcl} F_n : \bar{\mathbb{R}} & \to & [0,1] \\ x & \mapsto & \displaystyle\frac{1}{n}\sum_{i=1}^{n} 1_{\{X_i \leq x\}}. \end{array}$$

Show that,

$$\sup_{x\in\mathbb{R}} |F_n(x) - F(x)| \overset{n\to\infty}{\longrightarrow} 0 \quad P\text{ -a.s.}$$

Hint: Show as an intermediate step that for every continuous and non-decreasing function $F : \bar{\mathbb{R}} \to [0,1]$ and every sequence $(F_n : \bar{\mathbb{R}} \to [0,1])_{n\geq 1}$ of non-decreasing functions it holds that if $F_n(x) \overset{n\to\infty}{\longrightarrow} F(x)$ for all $x \in \bar{\mathbb{Q}} = \mathbb{Q} \cup \{\pm\infty\}$, then $(F_n)_{n\geq 1}$ converge uniformly to F.
Remark: The statement of Glivenko-Cantelli also holds for non-continous distribution functions as well.

Exercise 4.2
Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables in a probability space $(\Omega, \mathcal{A}, P)$. Define the two sequences of random variables $(Y_n)_{n\geq 1}$ and $(M_n)_{n\geq 1}$ by

$$Y_n := \min_{1\leq i\leq n} X_i \quad \text{and} \quad M_n := \max_{1\leq i\leq n} X_i$$

1. Let X_1 be uniformly distributed on the interval $[0,1]$. Show that nY_n converges in distribution to an exponential random variable Z with parameter 1, i.e., the density of Z is $e^{-x}1_{[0,\infty)}(x), x \in \mathbb{R}$.
2. Let X_1 be exponentially distributed with parameter 1. Show that $M_n - \log n$ converges in distribution to a random variable Z with Gumbel distribution, i.e. the density of Z is $e^{-x}\exp(-e^{-x}), x \in \mathbb{R}$.

Exercise 4.3

(a) Let f be a (not necessarily Borel-measurable) function from $\mathbb{R}$ to $\mathbb{R}$. Show that the set of discontinuities of f, defined as

$$U_f := \left\{x \in \mathbb{R} \mid f \text{ is discontinuous in } x\right\},$$

is Borel-measurable.

(b) Assume that $X_n \to X$ in distribution. Let f be measurable and bounded, such that $P[X \in U_f] = 0$. Use (2.2.13) − (2.2.14) from the lecture notes to show that we have

$$E[f(X_n)] \underset{n\to\infty}{\to} E[f(X)].$$

(c) Let f be measurable and bounded on $[0,1]$, with U_f of Lebesgue measure 0. Show that the corresponding Riemann sums converge to the integral of f, i.e.

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) \underset{n\to\infty}{\to} \int_0^1 f(x)\mathrm{d}x.$$

Submission: until 12:00, Oct 20., through the webpage of the course. You should carefully follow the **submission instructions** on the webpage to get your solutions back.

Office hours: See the webpage for detailed information

- Präsenz (Group 3): Mon. and Thu., 12:00-13:00 in HG G32.6. with previous reservation.
- Probability Theory Assistants: Tue. 15:30-16:30 and Wed. 11:00-12:00 via Zoom with a 10 minutes slot reservation.

Exercise class: Online. In-person exercise classes need previous registration each week.

Exercise sheets and further information are also available on:
https://metaphor.ethz.ch/x/2020/hs/401-3601-00L/

Solution 4.1

We start proving the statement of the hint. Let $\epsilon > 0$ and take $-\infty = x_0 < x_1 < \cdots < x_n = \infty \in \bar{\mathbb{Q}} = \mathbb{Q} \cup \{\pm\infty\}$ for some $n \in \mathbb{N}$ such that

$$F(x_{i+1}) - F(x_i) \leq \epsilon, \tag{1}$$

note that this is possible by the continuity of F and the fact that F it is non-decreasing with $\lim_{x\to\infty} f(x) = 1$ and $\lim_{x\to-\infty} f(x) = 0$. We assume that $F_m(x) \overset{m\to\infty}{\to} F(x)$ for all $x \in \bar{\mathbb{Q}}$, which imply that there exists $N \in \mathbb{N}$ such that for all $m \geq N$

$$\sup_{0\leq i\leq n} |F_m(x_i) - F(x_i)| < \epsilon. \tag{2}$$

Note that for all $x \in \bar{\mathbb{R}}$ there exists $i \in \{0, \dots, n-1\}$ such that $x_i \leq x \leq x_{i+1}$. Combining (1) and (2) we get that,

$$F_m(x) - F(x) \leq F_m(x_{i+1}) - [F(x_{i+1}) - \epsilon] \leq 2\epsilon.$$

and

$$F(x) - F_m(x) \leq [F(x_i) + \epsilon] - F_m(x_i) \leq 2\epsilon.$$

Therefore $|F_m(x) - F(x)| \leq 2\epsilon$ for all $x \in \bar{\mathbb{R}}$ and $m \geq N$, so we get the desired uniform convergence. Now we apply this result to our problem. Let us remark that for each given $x \in \bar{\mathbb{R}}$, $F_n(x)$ is actually a random variable: $\omega \mapsto \frac{1}{n}\sum_{i=1}^n 1_{\{X_i(\omega)\leq x\}}$. To keep notation clean we usually omit ω in F_n.
By the Strong Law of Large Numbers,

$$F_n(x) = \frac{1}{n}\sum_{i=1}^n 1_{\{X_i\leq x\}} \to P(X_i \leq x) = F(x),\ P\text{-a.s.},$$

which imply that for all $x \in \bar{\mathbb{Q}}$ ($\pm\infty$ is trivial) there exists $N_x \subset \Omega$ with $P(N_x) = 0$ such that for all $\omega \notin N_x$, $F_n(x,\omega) \to F(x)$ as n goes to ∞. Let N be defined as $N := \bigcup_{y\in\bar{\mathbb{Q}}} N_y$, then for all $x \in \bar{\mathbb{Q}}$ and $\omega \in N$, $F_n(x,\omega) \to F(x)$ pointwise, therefore, by the hint above, $\forall\omega \notin N$, $F_n \to F$ uniformly as n goes to infinity. Since N is the countable union of set with measure 0 we have that $P(N) = 0$ and then $F_n(x) \to F(x)$ uniformly P-a.s..

Solution 4.2

1. Take $y \in \mathbb{R}$. For $y < 0$, because the exponential distribution is concentrated on $[0,\infty)$, we have that

$$P[nY_n \leq y] = 0 = P[Z \leq y] \text{ for all } n \in \mathbb{N}.$$

Hence without loss of generality we assume $y \geq 0$. It follows that for all $n \geq y$:

$$\begin{aligned} P\left[nY_n \leq y\right] &= P\left[Y_n \leq \frac{y}{n}\right] = P\left[\bigcup_{i=1}^n \left\{X_i \leq \frac{y}{n}\right\}\right] = 1 - P\left[\bigcap_{i=1}^n \left\{X_i > \frac{y}{n}\right\}\right] \\ &= 1 - \prod_{i=1}^n P\left[X_i > \frac{y}{n}\right] = 1 - \left(1 - \frac{y}{n}\right)^n. \end{aligned}$$

Now let $n \to \infty$, we hence obtain that

$$\lim_{n\to\infty} P[nY_n \leq y] = 1 - e^{-y} = \int_{-\infty}^y e^{-x} 1_{[0,\infty)}(x)\,\mathrm{d}x.$$

2. Take $y \in \mathbb{R}$. Then for all $n \geq e^{-y}$ we have that:

$$P\left[M_n - \log n \leq y\right] = P\left[M_n \leq y + \log n\right] = P\left[\bigcap_{i=1}^{n} \{X_i \leq y + \log n\}\right]$$
$$= \prod_{i=1}^{n} \underbrace{P\left[X_i \leq y + \log n\right]}_{=1-e^{-y-\log n}=1-\frac{e^{-y}}{n}} = \left(1 - \frac{e^{-y}}{n}\right)^n.$$

Now let $n \to \infty$, we hence obtain that

$$\lim_{n\to\infty} P\left[M_n - \log n \leq y\right] = \exp(-e^{-y}) = \int_{-\infty}^{y} e^{-x} \exp(-e^{-x})\,\mathrm{d}x.$$

Solution 4.3

(a) Let $f : \mathbb{R} \to \mathbb{R}$ and

$$V_{\epsilon,\delta} := \left\{x \in \mathbb{R} | \exists y, z \in (x-\delta, x+\delta) \text{ s.t. } |f(y) - f(z)| \geq \epsilon\right\}.$$

(i) Claim: $V_{\epsilon,\delta}$ is open.
Let $x \in V_{\epsilon,\delta}$. Then there are $y, z \in (x-\delta, x+\delta)$ such that $|f(y) - f(z)| \geq \epsilon$. We set $r := \delta - \max\left\{|y-x|, |z-x|\right\} > 0$.
$\Rightarrow \forall\ \tilde{x} \in (x-r, x+r)$ it holds that $|y-\tilde{x}| \leq |y-x| + |x-\tilde{x}| < |y-x| + r \leq \delta$, and similarly for z. From this it follows that $y, z \in (\tilde{x}-\delta, \tilde{x}+\delta)$ and $|f(y)-f(z)| \geq \epsilon$, which gives $\tilde{x} \in V_{\epsilon,\delta}$. So $(x-r, x+r) \subset V_{\epsilon,\delta}$, and the claim follows.

(ii) Claim: $U_f = \bigcup_n \bigcap_m V_{\frac{1}{n},\frac{1}{m}}$.
"$\subset$" Let $x \in U_f$. Then there is an $n \in \mathbb{N}$, such that

$$\forall\ m \in \mathbb{N}\ \exists\ y \in \left(x - \frac{1}{m}, x + \frac{1}{m}\right) \text{ s.t. } |f(x) - f(y)| \geq \frac{1}{n}.$$

"$\supset$" We assume that for some $n \in \mathbb{N}$, $x \in V_{\frac{1}{n},\frac{1}{m}}, \forall\ m$. Then there are $y, z \in (x - \frac{1}{m}, x + \frac{1}{m})$ so that $|f(y) - f(z)| \geq \frac{1}{n}$.
From this it follows that either $|f(y) - f(x)| \geq \frac{1}{2n}$ or $|f(z) - f(x)| \geq \frac{1}{2n}$ must hold. In other words $\exists\ n \in \mathbb{N}$, such that

$$\forall\ m \in \mathbb{N}\ \exists\ y \in \left(x - \frac{1}{m}, x + \frac{1}{m}\right) : |f(y) - f(x)| \geq \frac{1}{2n},$$

which implies that f is discontinuous in x.

Since the $V_{\frac{1}{n},\frac{1}{m}}$ are open, they are Borel measurable. And since any $\sigma-$algebra is closed under countable unions and intersections, $U_f = \bigcup_n \bigcap_m V_{\frac{1}{n},\frac{1}{m}}$ must also be Borel measurable.

(b) By (2.2.13) – (2.2.14) of the lecture notes, there exist $Y_n \overset{d}{=} X_n$, and $Y \overset{d}{=} X$, such that $Y_n \to Y$, P'-almost surely on a probability space $(\Omega', \mathcal{F}', P')$. Of course, we also have $f(Y_n) \overset{d}{=} f(X_n)$, and $f(Y) \overset{d}{=} f(X)$, so that we have $E[f(X_n)] = E'[f(Y_n)]$, and $E[f(X)] = E'[f(Y)]$, where we denote by E' the expectation w.r.t. P'. Thus, it suffices to show that

$$E'[f(Y_n)] \underset{n\to\infty}{\to} E'[f(Y)]. \tag{3}$$

Now, since $Y_n \to Y$, P'-almost surely, we have a set N, with $P'(N) = 0$, such that

$$\left\{\omega' \in \Omega' \mid Y_n(\omega') \underset{n\to\infty}{\to} Y(\omega')\right\} \cup N = \Omega'. \tag{4}$$

On the other hand, we have

$$\begin{aligned}\{\omega' \mid Y_n(\omega') \to Y(\omega')\} &\subseteq \{\omega' \mid f \text{ cont. in } Y(\omega'), Y_n(\omega') \to Y(\omega')\} \cup \{\omega \mid f \text{ discont. in } Y(\omega)\} \\ &\subseteq \{\omega' \mid f(Y_n(\omega')) \to f(Y(\omega'))\} \cup \{\omega' \mid Y(\omega') \in U_f\}.\end{aligned}$$

Consequently, it follows from equation (4) that we have

$$\{\omega' \mid f(Y_n(\omega')) \to f(Y(\omega'))\} \cup \{\omega' \mid Y(\omega') \in U_f\} \cup N = \Omega'.$$

But, by assumption $P'(Y \in U_f) = P(X \in U_f) = 0$ (recall that Y and X have the same distribution). Therefore, we get $f(Y_n) \to f(Y)$, P'-almost surely. Finally f is a bounded function, so by the Dominated Convergence Theorem equation (3) holds.

(c) Let λ be the Lebesgue measure on $[0,1]$, and, for all $a \in [0,1]$, let δ_a denote the Dirac delta measure on $[0,1]$. Let X_n, $n \geq 1$, be random variables with distribution $\frac{1}{n}\sum_{k=1}^{n} \delta_{\frac{k}{n}}$. Note that

$$E[f(X_n)] = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right).$$

Let X be a uniform random variable on $[0,1]$, hence it has distribution λ, and we note that

$$E[f(X)] = \int_0^1 f(x)\lambda(dx).$$

Thus, it suffices to show that we have

$$E[f(X_n)] \underset{n\to\infty}{\to} E[f(X)]. \tag{5}$$

Since by assumption $P[X \in U_f] = \lambda(U_f) = 0$, part **b)** implies that equation (5) is a consequence of the following:

$$X_n \overset{d}{\to} X. \tag{6}$$

To show equation (6), note that for all $n \in \mathbb{N}$,

$$P\left[X_n \leq a\right] = \begin{cases} 0, & a < 0, \\ \frac{[na]}{n}, & 0 \leq a \leq 1, \\ 1, & 1 < a. \end{cases}$$

Since we have $na - 1 < [na] \leq na$ (i.e. $[na]$ denotes the integer part of na), we get $\frac{[na]}{n} \underset{n\to\infty}{\to} a$, for all $0 \leq a \leq 1$. Thus, we obtain, for $0 \leq a \leq 1$,

$$P\left[X_n \leq a\right] \underset{n\to\infty}{\to} \lambda([0,a]) = P\left[X \leq a\right],$$

which implies equation (6), by definition. (Cases for $a < 0$ and $a > 1$ are trivially verified.)