ETH Zurich HS 2020
D-MATH, D-PHYS
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Coordinator
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Probability Theory

Exercise Sheet 5

Exercise 5.1

(a) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of real random variables converging in probability to a random variable X. Show that $(X_n)_{n\in\mathbb{N}}$ converges to X in distribution.

(b) The converse does not hold in general. Instead, show that if the sequence $(X_n)_{n\in\mathbb{N}}$ converges in distribution to a *constant* random variable $X = c$, then $(X_n)_{n\in\mathbb{N}}$ converges in probability to c.

Exercise 5.2 Compute the characteristic functions of the following distributions:

(a) The triangular distribution $(1-|x|)1_{[-1,1]}(x)\mathrm{d}x$.

(b) The Cauchy distribution $\frac{\alpha}{\pi}\frac{1}{x^2+\alpha^2}\mathrm{d}x$ with parameter $\alpha > 0$.
Hint: Use a contour integral.

Exercise 5.3

(a) Let μ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, with characteristic function ϕ, and let a be in $\mathbb{R}$. Show that we have

$$\frac{1}{2T}\int_{-T}^{T} e^{-ita}\phi(t)\mathrm{d}t \underset{T\to\infty}{\to} \mu(\{a\}).$$

Hint: Use Fubini's theorem, and the Dominated Convergence Theorem.

(b) Let X, Y be independent random variables, with distribution ν, and characteristic function ψ. Show that we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\psi(t)|^2\mathrm{d}t = P(X=Y) = \sum_{x\in\mathbb{R}}\nu(\{x\})^2,$$

where we sum over all $x \in \mathbb{R}$, with $\nu(\{x\}) > 0$, on the right-hand side. (Prove that there can be at most countably many such x.)

Hint: Apply part (a) to the distribution of $X - Y$.

Submission: until 12:00, Oct 27., through the webpage of the course. You should carefully follow the **submission instructions** on the webpage to get your solutions back.

Office hours: See the webpage for detailed information

- Präsenz (Group 3): Mon. and Thu., 12:00-13:00 in HG G32.6. with previous reservation.

- Probability Theory Assistants: Tue. 15:30-16:30 and Wed. 11:00-12:00 via Zoom with a 10 minutes slot reservation.

Exercise class: Online. In-person exercise classes need previous registration each week.

Exercise sheets and further information are also available on:
https://metaphor.ethz.ch/x/2020/hs/401-3601-00L/

Solution 5.1

(a) We denote the distribution functions of X_n and X by F_n and F respectively. We have to show that $\lim_{n\to\infty} F_n(y) = F(y)$ for all y continuity point of F.

Let $y \in \mathbb{R}$ be a continuity point of F and let $\varepsilon > 0$. By the continuity of F in y there is a $\delta > 0$ such that

$$F(y) - \varepsilon \leq F(x) \leq F(y) + \varepsilon, \quad x \in [y-\delta, y+\delta]. \tag{1}$$

Since the X_n converge to X in probability, there is a $N \in \mathbb{N}$ such that

$$P\left[|X_n - X| > \delta\right] \leq \varepsilon, \quad n \geq N. \tag{2}$$

Now, for $n \geq N$,

$$\begin{aligned} F_n(y) = P[X_n \leq y] &\leq P[\{X \leq y+\delta\} \cup \{|X - X_n| > \delta\}] \\ &\leq P[X \leq y+\delta] + P[|X - X_n| > \delta] \\ &\overset{(2)}{\leq} F(y+\delta) + \varepsilon \overset{(1)}{\leq} F(y) + 2\varepsilon \end{aligned}$$

and

$$\begin{aligned} F_n(y) = P[X_n \leq y] &\geq P[\{X \leq y-\delta\} \setminus \{|X - X_n| > \delta\}] \\ &\geq P[X \leq y-\delta] - P[|X - X_n| > \delta] \\ &\overset{(2)}{\geq} F(y-\delta) - \varepsilon \overset{(1)}{\geq} F(y) - 2\varepsilon, \end{aligned}$$

so that

$$F(y) - 2\varepsilon \leq F_n(y) \leq F(y) + 2\varepsilon.$$

Thus,

$$F(y) - 2\varepsilon \leq \liminf_{n\to\infty} F_n(y) \leq \limsup_{n\to\infty} F_n(y) \leq F(y) + 2\varepsilon.$$

But this holds for all $\varepsilon > 0$, so we are done.

(b) We assume that for a $c \in \mathbb{R}$

$$X_n \to c \text{ in distribution.} \tag{3}$$

The constant c has distribution function

$$F(x) = 1_{[c,\infty)},$$

which is continuous except in c. So we know from (3)

$$F_n(z) = P[X_n \leq z] \to \begin{cases} 0 & \text{if } z < c, \\ 1 & \text{if } z > c. \end{cases} \tag{4}$$

We want to show that for all $\varepsilon > 0$

$$\lim_{n\to\infty} P[|X_n - c| \geq \varepsilon] = 0.$$

Now,

$$P[|X_n - c| \geq \varepsilon] = P[X_n \leq c - \varepsilon] + P[X_n \geq c + \varepsilon],$$

so that

$$\lim_{n\to\infty} P[|X_n - c| \geq \varepsilon] \leq \lim_{n\to\infty} P[X_n \leq c - \varepsilon] + \lim_{n\to\infty} P[X_n \geq c + \varepsilon].$$

By (4), we have $\lim_{n\to\infty} P[X_n \leq c-\varepsilon] = 0$ for all $\varepsilon > 0$. Furthermore

$$P[X_n \geq c+\varepsilon] = 1 - P[X_n < c+\varepsilon] \leq 1 - P\left[X_n \leq c + \frac{\varepsilon}{2}\right].$$

Thus,

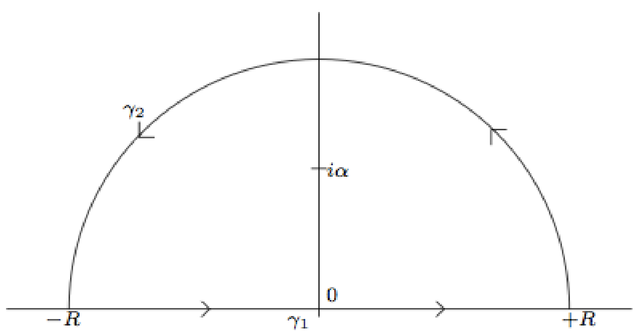
$$\lim_{n\to\infty} P[X_n \geq c+\varepsilon] \leq 1 - \lim_{n\to\infty} P\left[X_n \leq c + \frac{\varepsilon}{2}\right] \overset{(4)}{=} 0.$$

Solution 5.2

(a) We calculate

$$\begin{aligned} E[e^{itX}] &= \int_{\mathbb{R}} e^{itx}(1-|x|)1_{[-1,1]}(x)\mathrm{d}x = \int_{-1}^{1}(1-|x|)e^{itx}\mathrm{d}x \\ &= \int_{-1}^{1} e^{itx}\mathrm{d}x + \int_{-1}^{0} xe^{itx}\mathrm{d}x - \int_{0}^{1} xe^{itx}\mathrm{d}x \\ &= \frac{1}{it}\left(e^{it}-e^{-it}\right) + \left(\frac{e^{-it}}{it} + \frac{1}{t^2} - \frac{e^{-it}}{t^2}\right) - \left(\frac{e^{it}}{it} + \frac{e^{it}}{t^2} - \frac{1}{t^2}\right) \\ &= \frac{2}{t}\sin(t) - \frac{2}{t}\sin(t) - \frac{2}{t^2}\cos(t) + \frac{2}{t^2} = \frac{2}{t^2}(1-\cos(t)). \end{aligned}$$

(b) We use the Residue Theorem (for more details on Residue theorem and contour integrals we refer you to the book “Real and complex analysis” by Rudin) for $f(x) = \frac{e^{itx}}{x^2+\alpha^2}$ to calculate $E\left[e^{itX}\right] = \frac{\alpha}{\pi}\int_{-\infty}^{+\infty}\frac{e^{itx}}{x^2+\alpha^2}\mathrm{d}x$. Extend f to $\mathbb{C}$ by $f(z) = \frac{e^{itz}}{z^2+\alpha^2}$. Let us consider the curve $\gamma = \gamma_1 \cup \gamma_2$ as follows:



Thus, we calculate for all $R > \alpha$,

$$\oint_\gamma f(z)\mathrm{d}z = 2\pi i \cdot \underbrace{n(\gamma, i\alpha)}_{=1} \cdot \mathrm{Res}_{i\alpha}(f) = 2\pi i \cdot \lim_{z\to i\alpha}(z-i\alpha)f(z) = 2\pi i \cdot \lim_{z\to i\alpha}\frac{e^{itz}}{z+i\alpha} = \frac{\pi}{\alpha}e^{-\alpha t}.$$

Since we have, for $t \geq 0$,

$$\left|\int_{\gamma_2} f(z)\mathrm{d}z\right| = \left|\int_0^\pi f\left(Re^{i\theta}\right) iRe^{i\theta}\mathrm{d}\theta\right| = \left|\int_0^\pi \frac{e^{itRe^{i\theta}} iRe^{i\theta}}{R^2e^{2i\theta}+\alpha^2}\mathrm{d}\theta\right|$$
$$\leq \int_0^\pi \left|\frac{e^{itRe^{i\theta}} iRe^{i\theta}}{R^2e^{2i\theta}+\alpha^2}\right|\mathrm{d}\theta = \int_0^\pi \frac{Re^{-Rt\sin(\theta)}}{|R^2e^{2i\theta}+\alpha^2|}\mathrm{d}\theta$$
$$\leq \int_0^\pi \frac{R}{|R^2e^{2i\theta}+\alpha^2|}\mathrm{d}\theta \leq \int_0^\pi \frac{R}{R^2-\alpha^2}\mathrm{d}\theta = \frac{\pi R}{R^2-\alpha^2} \underset{R\to\infty}{\to} 0,$$

we obtain that

$$E\left[e^{itX}\right] = \lim_{R\to\infty}\frac{\alpha}{\pi}\int_{\gamma_1} f(z)\mathrm{d}z = \lim_{R\to\infty}\frac{\alpha}{\pi}\int_{\gamma} f(z)\mathrm{d}z = e^{-\alpha t}.$$

For $t<0$, one can use a similar argument to show $E\left[e^{itX}\right] = e^{\alpha t}$. Thus, we get, for $t\in\mathbb{R}$,

$$E\left[e^{itX}\right] = e^{-\alpha|t|}.$$

Solution 5.3

(a) We calculate

$$\begin{aligned}\frac{1}{2T}\int_{-T}^{T} e^{-ita}\phi(t)\mathrm{d}t &= \frac{1}{2T}\int_{-T}^{T}\int_{\mathbb{R}} e^{it(x-a)}\mu(\mathrm{d}x)\mathrm{d}t \\ &\overset{\text{Fub.}}{=} \frac{1}{2T}\int_{\mathbb{R}}\int_{-T}^{T} e^{it(x-a)}\mathrm{d}t\mu(\mathrm{d}x) = \int_{\mathbb{R}} f_a^T(x)\mu(\mathrm{d}x),\end{aligned}$$

where we have

$$f_a^T(x) := \begin{cases}\frac{\sin(T(x-a))}{T(x-a)}, & x\neq a,\\ 1, & x=a.\end{cases}$$

Now, f_a^T is bounded uniformly in T, (recall that $\lim_{x\to 0}\sin(x)/x = 1$, and consequently $\sin(x)/x$ is bounded). Also, we have

$$\lim_{T\to\infty} f_a^T(x) = \begin{cases}0, & x\neq a\\ 1, & x=a\end{cases} = 1_{\{a\}}(x)$$

Hence, applying the Dominated Convergence Theorem, yields the claim.

(b) We have $\psi(t) = E\left[e^{itX}\right] = E\left[e^{itY}\right]$. Therefore, we obtain

$$\begin{aligned}\frac{1}{2T}\int_{-T}^{T}\left|\psi(t)\right|^2\mathrm{d}t &= \frac{1}{2T}\int_{-T}^{T}\psi(t)\overline{\psi(t)}\mathrm{d}t = \frac{1}{2T}\int_{-T}^{T} E\left[e^{itX}\right]E\left[e^{-itY}\right]\mathrm{d}t \\ &\overset{\text{indep.}}{=} \frac{1}{2T}\int_{-T}^{T} E\left[e^{it(X-Y)}\right]\mathrm{d}t = \frac{1}{2T}\int_{-T}^{T} e^{-it\cdot 0}\phi_{X-Y}(t)\mathrm{d}t \\ &\xrightarrow[T\to\infty]{} P[X-Y=0] = P[X=Y].\end{aligned}$$

Furthermore, we have

$$\begin{aligned}P[X=Y] &= \nu\otimes\nu\left(\left\{(x,y)\in\mathbb{R}^2\middle| x=y\right\}\right) = \int_{\mathbb{R}^2} 1_{\{x=y\}}\nu(\mathrm{d}x)\nu(\mathrm{d}y) \\ &\overset{\text{Fub.}}{=} \int_{\mathbb{R}}\nu(\{x\})\nu(\mathrm{d}x) = \int_{\{x|\nu(\{x\})>0\}}\nu(\{x\})\nu(\mathrm{d}x) = \sum_{x\in\mathbb{R}}\nu(\{x\})^2,\end{aligned}$$

since we have $\{x \mid \nu(\{x\}) > 0\} = \bigcup_{n\in\mathbb{N}} A_n$, with $A_n = \{x \mid \nu(\{x\}) \geq \frac{1}{n}\}$, which implies $\{x \mid \nu(\{x\}) > 0\}$ is countable, as the A_n's are finite because $|A_n|/n \leq \sum_{x\in A_n} \nu(\{x\}) \leq 1$.