ETH Zurich HS 2020
D-MATH, D-PHYS
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Probability Theory

Exercise Sheet 6

Exercise 6.1 Let $(X_i)_{i\geq 1}$ be i.i.d. with symmetric stable distribution of parameter $\alpha \in (0,2)$, see lecture notes p. 63.

(a) Find the distribution of $n^{-1/\alpha}(X_1 + \cdots + X_n)$.

(b) Does $\frac{1}{\sqrt{n}}(X_1 + \cdots + X_n)$ converge in distribution?

Exercise 6.2 Let $X_1, X_2, \ldots$ be independent random variables for which there exists a constant $M > 0$, such that $|X_n| \leq M$, P-a.s. for $n = 1, 2, \ldots$. We write $S_n = X_1 + \ldots + X_n$. Show that, if $\sum \mathrm{Var}(X_n) = \infty$, then there exist constants a_n, b_n such that $(S_n - b_n)/a_n$ converges in distribution towards a standard normal random variable.

Hint: Use the Lindeberg-Feller theorem (Theorem 2.24, p. 71 of the lecture notes).

Exercise 6.3 Show that when Y_k, $k \geq 1$ are independent uniformly bounded random variables such that $\sum_k Y_k$ converges P-a.s., then $\sum_k \mathrm{Var}(Y_k) < \infty$.

Hint: consider independent copies $\tilde{Y}_k$, $k \geq 1$ of the Y_k, $k \geq 1$ and use Exercise 6.2 with $X_k = Y_k - \tilde{Y}_k$, $k \geq 1$.

Submission: until 12:00, Nov. 3, through the webpage of the course. You should carefully follow the **submission instructions** on the webpage to get your solutions back.

Office hours: See the webpage for detailed information

- Präsenz (Group 3): Mon. and Thu., 12:00-13:00 in HG G32.6. with previous reservation.
- Probability Theory Assistants: Tue. 15:30-16:30 and Wed. 11:00-12:00 via Zoom with a 10 minutes slot reservation.

Exercise class: Online. In-person exercise classes need previous registration each week.

Exercise sheets and further information are also available on:
https://metaphor.ethz.ch/x/2020/hs/401-3601-00L/

Solution 6.1 Let $S_n = \sum_{i=1}^n X_i$.

(a) Note that $\frac{1}{n^{1/\alpha}}(X_1 + \cdots + X_n) = n^{-1/\alpha} S_n$. Using that the random variables are i.i.d. and that the characteristic function is given by $\varphi_{X_1}(t) = \exp(-c|t|^\alpha)$ with $c > 0$,

$$\begin{aligned}\varphi_{\frac{S_n}{n^{1/\alpha}}}(t) = \varphi_{S_n}(t/n^{1/\alpha}) &= \prod_{i=1}^n \varphi_{X_i}(t/n^{1/\alpha}) = \varphi_{X_1}(t/n^{1/\alpha})^n \\ &= \left(e^{-c|t|^\alpha/n}\right)^n = e^{-c|t|^\alpha} = \varphi_{X_1}(t),\end{aligned}$$

showing that $\frac{1}{n^{1/\alpha}} S_n$ is distributed as X_1.

(b) Note that

$$\frac{X_1 + \cdots + X_n}{\sqrt{n}} = \frac{S_n}{n^{1/\alpha}} \frac{n^{1/\alpha}}{\sqrt{n}}.$$

By (a),

$$\varphi_{\frac{S_n}{n^{1/\alpha}} \frac{n^{1/\alpha}}{\sqrt{n}}}(t) = \varphi_{\frac{S_n}{n^{1/\alpha}}}\left(\frac{n^{1/\alpha}}{\sqrt{n}} t\right) = \varphi_{X_1}\left(\frac{n^{1/\alpha}}{\sqrt{n}} t\right).$$

Since $\alpha \in (0, 2)$,

$$\lim_{n\to\infty} \varphi_{\frac{S_n}{n^{1/\alpha}}}\left(\frac{n^{1/\alpha}}{\sqrt{n}} t\right) = \lim_{n\to\infty} \varphi_{X_1}\left(\frac{n^{1/\alpha}}{\sqrt{n}} t\right) = \lim_{n\to\infty} \exp(-c|n^{1/\alpha - 1/2} t|^\alpha) = \begin{cases} 1, & \text{if } t = 0, \\ 0, & \text{otherwise,} \end{cases}$$

which, since it is not continuous, is not the characteristic function of any distribution. Hence, by the contrapositive of (2.3.24) from the lecture notes,

$$\frac{X_1 + \cdots + X_n}{\sqrt{n}}$$

does not converge in distribution.

Solution 6.2 We use the Lindeberg-Feller theorem (Theorem 2.24, p. 71 in lecture notes). We define

$$Y_{n,i} = \frac{X_i - E[X_i]}{\sqrt{\sum_{j=1}^n \mathrm{Var}(X_j)}}, \quad i = 1, \ldots, n$$

(For the finitely many n where possibly $\sum_{j=1}^n \mathrm{Var}(X_j) = 0$, we set $Y_{n,i} \equiv 0$). Then it follows that

$$\sum_{i=1}^n E[Y_{n,i}^2] \xrightarrow{n\to\infty} 1.$$

More precisely, except for the finitely many n mentioned above,

$$\sum_{i=1}^n E[Y_{n,i}^2] = \sum_{i=1}^n \frac{E[(X_i - E[X_i])^2]}{\sum_{j=1}^n \mathrm{Var}(X_j)} = \frac{\sum_{i=1}^n \mathrm{Var}(X_i)}{\sum_{j=1}^n \mathrm{Var}(X_j)} = 1,$$

which justifies the first condition.

We now verify the second condition. For $\epsilon > 0$ we take $n_0 \in \mathbb{N}$ such that

$$\sum_{j=1}^n \mathrm{Var}(X_j) \geq \frac{(2M)^2}{\epsilon^2}, \quad \forall n \geq n_0,$$

which exists since $\sum \operatorname{Var}(X_j) = \infty$. Then by the fact that $|X_i|$ are uniformly bounded by M, one has

$$|Y_{n,i}| = \left| \frac{X_i - E[X_i]}{\sqrt{\sum_{j=1}^n \operatorname{Var}(X_j)}} \right| \leq \frac{2M}{2M/\epsilon} \leq \epsilon$$

for $n \geq n_0$. Hence,

$$1_{\{|Y_{n,i}|>\epsilon\}} \equiv 0, \ \forall n \geq n_0, \forall i \leq n.$$

In fact,

$$\lim_{n\to\infty} \sum_{i=1}^n E\left[Y_{n,i}^2 1_{\{|Y_{n,i}|>\epsilon\}}\right] = 0.$$

Therefore all the conditions are fulfilled, whence

$$\sum_{i=1}^n Y_{n,i} \xrightarrow{n\to\infty} \mathcal{N}(0,1) \text{ in distribution.}$$

On the other hand we can rewrite $\sum_{i=1}^n Y_{n,i}$ as

$$\sum_{i=1}^n Y_{n,i} = \frac{\sum_{i=1}^n (X_i - E[X_i])}{\sqrt{\sum_{j=1}^n \operatorname{Var}(X_j)}},$$

and the claim follows with

$$a_n := \sqrt{\sum_{j=1}^n \operatorname{Var}(X_j)}, \qquad b_n := E\left[\sum_{j=1}^n X_j\right].$$

Solution 6.3 Let $\tilde{Y}_k$, $k \geq 1$ be independent copies of Y_k, $k \geq 1$ and set $X_k := Y_k - \tilde{Y}_k$ as in the hint. Then X_k, $k \geq 1$ are independent uniformly bounded variables. Suppose to the contrary that $\sum_k Y_k$ converges P-a.s. but $\sum_k \operatorname{Var}(Y_k) = \infty$. Then also $S_n := X_1 + \cdots + X_n$ converges P-a.s. as a sum of two P-a.s. convergent sequences $\sum_k Y_k$ and $\sum_k \tilde{Y}_k$, and $\sum_k \operatorname{Var}(X_k) = 2\sum_k \operatorname{Var}(Y_k) = \infty$. Define

$$a_n := \sqrt{\sum_{j=1}^n \operatorname{Var}(X_j)}, \qquad b_n := E\left[\sum_{j=1}^n X_j\right] = 0.$$

Then it follows as in the solution of Exercise 6.3 that $(S_n - b_n)/a_n = S_n/a_n$ converges in distribution towards a standard normal random variable. Since S_n is P-a.s. convergent, for each $\epsilon > 0$ there is a $N \in \mathbb{N}$ and a $M > 0$ such that for all $n \geq N$, $P[|S_n| \geq M] < \epsilon$. Since the sequence $(a_n)_n$ is monotone increasing towards infinity, we can find a $\tilde{N}$ such that for all $n \geq \tilde{N}$, $a_n \geq M$. Then for all $n \geq N \vee \tilde{N}$, $P[|S_n/a_n| \geq 1] \leq P[|S_n/a_n| \geq M/a_n] < \epsilon$. Since this can be done for any $\epsilon > 0$, S_n/a_n can not converge in distribution to a standard normal random variable, which is a contradiction.