ETH Zurich HS 2020
D-MATH, D-PHYS
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Probability Theory

Exercise Sheet 7

Exercise 7.1 Let X and Y be two independent Bernoulli distributed random variables with parameter p. Define $Z = 1_{\{X+Y=0\}}$ and $\mathcal{G} = \sigma(Z)$. Find $E[X|\mathcal{G}]$ and $E[Y|\mathcal{G}]$. Are these random variables also independent?

Exercise 7.2 Let Y and Z be independent random variables on $(\Omega, \mathcal{A}, P)$ with respective distributions μ and ν, and $f : \mathbb{R}^2 \to \mathbb{R}$ be a bounded measurable function. Let $X = f(Y, Z)$, and $h : \mathbb{R} \to \mathbb{R}$ be the bounded measurable function

$$h(y) = \int_{\mathbb{R}} f(y, z) d\nu(z), \text{ for } y \in \mathbb{R}.$$

Show that $E[X|\sigma(Y)] = h(Y)$ P-a.s.

Exercise 7.3 Let S be a random variable with $P[S > t] = e^{-t}$, for all $t > 0$. Calculate the conditional expectation $E\big[S \,\big|S \wedge t\big]$, where $S \wedge t := \min(S, t)$ for arbitrary $t > 0$.

Remark: Recall that by definition $E[X|Y] := E[X|\sigma(Y)]$ when X and Y are random variables in the same probability space and X is integrable.

Exercise 7.4 (Optional.) In this exercise we prove that in Theorem 1.37 (Kolmogorov's Three Series Theorem) (1.4.16) $\Rightarrow$ (1.4.17).

Consider X_k, $k \geq 1$ independent random variables and $A > 0$. Set $Y_k := X_k 1_{\{|X_k| \leq A\}}$, $k \geq 1$. Assume that $\sum_k X_k$ converges P-a.s.

(a) Show that $P[\liminf_k \{X_k = Y_k\}] = 1$.

(b) Deduce from (a) that $\sum_k P[|X_k| > A] < \infty$ and $\sum_k Y_k$ converges P-a.s.

(c) Show that $\sum_k \mathrm{Var}(Y_k) < \infty$. (**Hint:** use Exercise 6.3.)

(d) Show that $\sum_k E[Y_k]$ converges. (**Hint:** use Theorem 1.34, moreover (c) and (b).)

Submission: until 12:00, Nov. 10, through the webpage of the course. You should carefully follow the **submission instructions** on the webpage to get your solutions back.

Office hours: See the webpage for detailed information

- Präsenz (Group 3): Mon. and Thu., 12:00-13:00 in HG G32.6. with previous reservation.
- Probability Theory Assistants: Tue. 15:30-16:30 and Wed. 11:00-12:00 via Zoom with a 10 minutes slot reservation.

Exercise class: Online. In-person exercise classes need previous registration each week.

Exercise sheets and further information are also available on:
https://metaphor.ethz.ch/x/2020/hs/401-3601-00L/

Solution 7.1 Since Z is constant on each the sets $A_0 = \{X + Y = 0\}$ and $A_1 = \{X + Y \geq 1\}$, we know that $\mathcal{G}$ is generated by this partition. Thus,

$$E[X|\mathcal{G}](\omega) = \alpha_i = \frac{E[X1_{A_i}]}{P(A_i)}, \quad \text{for } \omega \in A_i.$$

On A_0, X is identically 0, so $E[X1_{A_0}] = 0$ and $\alpha_0 = 0$. On the other hand, $X1_{A_1} = 1_{\{X=1\}}1_{\{X+Y\geq 1\}} = 1_{\{X=1\}}$, so it follows that

$$\alpha_1 = \frac{p}{P(A_1)} = \frac{p}{1-(1-p)^2} = \frac{1}{2-p}.$$

Hence, the conditional expectation can be expressed as

$$E[X|\mathcal{G}] = \frac{1}{2-p}1_{\{X+Y\geq 1\}}.$$

By symmetry, $E[Y|\mathcal{G}]$ is given by the same expression, whence we conclude that $E[X|\mathcal{G}] = E[Y|\mathcal{G}]$. Since a non-constant random variable cannot be independent from itself, the two random variables $E[X|\mathcal{G}]$ and $E[Y|\mathcal{G}]$ are not independent.

Solution 7.2 Let us recall that $\sigma(Y) = \{Y^{-1}(A); A \in \mathcal{B}(\mathbb{R})\}$. In particular, $\sigma(Y)$ is a sub-σ-algebra of $\mathcal{A}$ and $h(Y)$ is $\sigma(Y)$-measurable (See Exercise 1.3). Since h is bounded, we also have that $h(Y)$ is integrable. This means that $h(Y)$ satisfies (3.1.12). Now we want to prove that it also satisfies (3.1.13) in the definition of conditional expectation. Let $A \in \mathcal{B}(\mathbb{R})$. Using (1.2.17) in the lecture notes and Fubini's theorem, we have

$$E[h(Y)1_{\{Y\in A\}}] = \int_{\mathbb{R}} h(y)1_A(y)d\mu(y) = \int_{\mathbb{R}^2} f(y,z)1_A(y)d\mu(y)d\nu(z).$$

By (1.3.13), we can rewrite the expression above and we obtain

$$E[h(Y)1_{\{Y\in A\}}] = E[f(Y,Z)1_{\{Y\in A\}}],$$

which is exactly (3.1.13) for $\mathcal{F} = \sigma(Y)$. Therefore, by P-a.s. uniqueness of the conditional expectation, the statement follows.

Solution 7.3 One has that

$$\begin{aligned} E\big[S \,\big|S \wedge t\big] &= E\big[S1_{\{S<t\}} \,\big|S \wedge t\big] + E\big[S1_{\{S\geq t\}} \,\big|S \wedge t\big] \\ &= E\big[(S\wedge t)1_{\{S\wedge t<t\}} \,\big|S \wedge t\big] + E\big[S1_{\{S\geq t\}} \,\big|S \wedge t\big] \\ &= (S\wedge t)1_{\{S\wedge t<t\}} + E\big[S1_{\{S\geq t\}} \,\big|S \wedge t\big]. \end{aligned} \tag{1}$$

We now compute the second term. Take arbitrary $A \in \mathcal{B}(\mathbb{R})$. Then one has that:

$$\begin{aligned} E\big[S1_{\{S\geq t\}}1_{\{S\wedge t\in A\}}\big] &= E\big[S1_{\{S\geq t\}}1_{\{t\in A\}}\big] = 1_{\{t\in A\}}\int_t^\infty xe^{-x}dx \\ &= 1_{\{t\in A\}}\Big[(-xe^{-x})\Big|_t^\infty + \int_t^\infty e^{-x}dx\Big] = 1_{\{t\in A\}}(t+1)e^{-t} \\ &= 1_{\{t\in A\}}(t+1)E\big[1_{\{S\geq t\}}\big] = E\big[(t+1)1_{\{S\geq t\}}1_{\{S\wedge t\in A\}}\big]. \end{aligned}$$

Because $\{S \geq t\} = \{S \wedge t = t\}$ is $\sigma(S \wedge t)$-measurable, we have that

$$E\big[S1_{\{S\geq t\}} \,\big|S \wedge t\big] = (t+1)1_{\{S\wedge t=t\}}.$$

Then it follows from (1) that

$$E\big[S \,\big|S \wedge t\big] = (S\wedge t)1_{\{S\wedge t<t\}} + (t+1)1_{\{S\wedge t=t\}}.$$

Solution 7.4

(a) We are going to argue by contraposition. Assume that $P[\liminf_k \{X_k = Y_k\}] < 1$, i.e. $P(A := \Omega \backslash \liminf_k \{X_k = Y_k\}) > 0$. Then by the definition of $\liminf$, for every $\omega \in A$ and $n \in \mathbb{N}$ there exists a $k > n$ such that $X_k(\omega) \neq Y_k(\omega)$. By the definition of Y this means that $|X_k(\omega)| > A$ for such a k. This implies by Cauchy's convergence test that $\sum_k X_k(\omega)$ does not converge. Since this is true for each $\omega \in A$ and $P(A) > 0$, it follows that $\sum_k X_k$ does not converge P-a.s.

(b) First, note that $\{|X_k| > A\} = \{X_k \neq Y_k\}$. Second, by de Morgan's law we have $\limsup_k \{X_k = Y_k\}^c = (\liminf_k \{X_k = Y_k\})^c$. Putting these together, we obtain due to (a) the following:

$$\begin{aligned} P(\limsup_k \{|X_k| > A\}) &= P(\limsup_k \{X_k \neq Y_k\}) = P(\limsup_k \{X_k = Y_k\}^c) \\ &= P((\liminf_k \{X_k = Y_k\})^c) = 1 - P(\liminf_k \{X_k = Y_k\}) = 0. \end{aligned} \tag{2}$$

Now since X_k are independent, so are the events $\{|X_k| > A\}$, and hence the contraposition of the second lemma of Borel-Cantelli (Lemma 1.26 in lecture notes) implies that $\sum_k P[|X_k| > A] < \infty$.

For the second statement, observe that $\sum_k Y_k = \sum_k X_k - \sum_k X_k 1_{|X_k|>A}$. Due to (2) we have $P(\limsup_k \{|X_k| > A\}) = 0$. This means that there is a set $A \in \mathcal{F}$ with $P(A) = 1$ such that for each $\omega \in A$ there exists a N such that for all $k > N$, $\omega \notin \{|X_k(\omega)| > A\}$, i.e. $1_{|X_k|>A}(\omega) = 0$. This in particular implies that $\sum_k X_k(\omega) 1_{|X_k|>A}(\omega)$ converges as the sum has only finitely many non-zero summands. Sine this is true for all $\omega \in A$ and $P(A) = 1$, it follows that $\sum_k X_k 1_{|X_k|>A}$ converges P-a.s. Therefore $\sum_k Y_k$ is a sum of two P-a.s convergent series and hence it converges P-a.s. itself.

(c) This is a direct consequence of Exercise 6.3. Indeed, Y_k are independent as X_k are, and uniformly bounded by A by construction. Now since in (b) we have shown that $\sum_k Y_k$ converges P-a.s., Exercise 6.3 implies that $\sum_k \mathrm{Var}(Y_k) < \infty$ and hence we are done.

(d) Define $Z_k := Y_k - E[Y_k]$. Then Z_k are independent as X_k and hence Y_k are, $\sum_k \mathrm{Var}(Z_k) = \sum_k \mathrm{Var}(Y_k) < \infty$ due to (c) and $E[Z_k] = 0$ for each k by construction. Hence the conditions of Theorem 1.34 in the lecture notes are satisfied and it follows that $\sum_k Z_k$ converges P-a.s. But since $\sum_k E[Y_k] = \sum_k Y_k - \sum Z_k$ and $\sum_k Y_k$ converges P-a.s. by (b), it follows that $\sum_k E[Y_k]$ also converges P-a.s. Since of course $E[Y_k]$ is deterministic for each k, it follows that $\sum_k E[Y_k]$ converges (for each $\omega \in \Omega$).