Probability Theory

Exercise Sheet 8

Exercise 8.1 Let X be a random variable in $L^2(\Omega, \mathcal{A}, P)$ and $\mathcal{F} \subseteq \mathcal{A}$. The *conditional variance* of X given $\mathcal{F}$ is defined as $\mathrm{Var}[X|\mathcal{F}] := E[(X - E[X|\mathcal{F}])^2|\mathcal{F}]$. Prove that

(a) $\mathrm{Var}[X|\mathcal{F}] = E[X^2|\mathcal{F}] - E[X|\mathcal{F}]^2$;

(b) $\mathrm{Var}(X) = E[\mathrm{Var}[X|\mathcal{F}]] + \mathrm{Var}[E[X|\mathcal{F}]]$.

(c) Compute $\mathrm{Var}[X|\mathcal{F}]$, where $\mathcal{F} = \sigma(A_1, A_2)$ where $\{A_1, A_2\}$ is a partition of Ω and $P(A_i) > 0$ for $i = 1, 2$.

Exercise 8.2 Let $n \geq 2$, and let $X_1, \ldots, X_n$ be i.i.d. random variables defined on a probability space $(\Omega, \mathcal{A}, P)$.

(a) Show that for every Borel function $g : \mathbb{R}^n \to \mathbb{R}$ with $E[|g(X_1, \ldots, X_n)|] < \infty$ and any permutation π of $\{1, \ldots, n\}$,

$$E[g(X_1, \ldots, X_n)] = E[g(X_{\pi(1)}, \ldots, X_{\pi(n)})].$$

(b) Set $S := X_1 + \ldots + X_n$ and assume that X_1 is integrable. Find a representation of $E[X_1|S]$ as a function of S.

Hint: First show that $E[X_1|S] = E[X_2|S]$ P-a.s.

Exercise 8.3 (Polya's Urn)
An urn initially contains s black and w white balls. We consider the following process. At each step a random ball is drawn from the urn, and is replaced by t balls of the same colour, for some fixed $t \geq 1$. We define the random variable Y_n as the proportion of black balls in the urn after the n-th iteration. Show that $E[Y_{n+1}|\sigma(Y_0, Y_1, \ldots Y_n)] = Y_n$, for all $n \geq 0$, that is, $\{Y_n\}_{n\geq 0}$ is a martingale.

Submission: until 12:00, Nov. 17, through the webpage of the course. You should carefully follow the **submission instructions** on the webpage to get your solutions back.

Office hours: Tue. 15:30-16:30 and Wed. 11:00-12:00 via Zoom with a 10 minutes slot reservation. Organized by the Probability Theory assistants.

Exercise class: Online. Details can be found on the polybox folder of the course.

Exercise sheets and further information are also available on:
https://metaphor.ethz.ch/x/2020/hs/401-3601-00L/

Solution 8.1

(a) Expanding the square in the definition, we obtain

$$\begin{aligned}\mathrm{Var}[X|\mathcal{F}] &:= E[(X - E[X|\mathcal{F}])^2|\mathcal{F}] \\ &= E[X^2 - 2XE[X|\mathcal{F}] + E[X|\mathcal{F}]^2|\mathcal{F}] \\ &= E[X^2|\mathcal{F}] - 2E[X|\mathcal{F}]E[X|\mathcal{F}] + E[X|\mathcal{F}]^2 \\ &= E[X^2|\mathcal{F}] - E[X|\mathcal{F}]^2.\end{aligned}$$

(b) Using the tower property and (a),

$$\begin{aligned}E[\mathrm{Var}[X|\mathcal{F}]] + \mathrm{Var}[E[X|\mathcal{F}]] &= E[E[X^2|\mathcal{F}] - E[X|\mathcal{F}]^2] + \mathrm{Var}[E[X|\mathcal{F}]] \\ &= E[X^2] - E[E[X|\mathcal{F}]^2] + E[E[X|\mathcal{F}]^2] - E[E[X|\mathcal{F}]]^2 \\ &= E[X^2] - E[X]^2 = \mathrm{Var}(X).\end{aligned}$$

(c) Using (a),

$$\begin{aligned}\mathrm{Var}[X|\mathcal{F}] &= \sum_{i=1}^{2} 1_{A_i} \left(\frac{E[X^2 1_{A_i}]}{P(A_i)} - \frac{E[X 1_{A_i}]^2}{P(A_i)^2} \right) \\ &= \sum_{i=1}^{2} 1_{A_i} \left(E[X^2|A_i] - E[X|A_i]^2 \right).\end{aligned}$$

Solution 8.2 Let Q_1 be the distribution of $(X_1, \ldots, X_n)$ and Q_2 the distribution of $(X_{\pi(1)}, \ldots, X_{\pi(n)})$.

(a) We claim that $Q_1 = Q_2$ over $\mathcal{B}(\mathbb{R}^n)$. To show this, by a consequence of Dynkin's lemma (see (1.3.11), p. 18 in lecture notes), it suffices to prove that

$$Q_1[A_1 \times \ldots \times A_n] = Q_2[A_1 \times \ldots \times A_n] \quad \text{with } A_j \in \mathcal{B}(\mathbb{R}).$$

By the definion of X_i,

$$\begin{aligned}P[X_1 \in A_1, \ldots, X_n \in A_n] &\overset{\text{i.i.d.}}{=} \prod_{j=1}^{n} P[X_j \in A_j] \\ &= \prod_{j=1}^{n} P[X_{\pi(j)} \in A_j] \\ &= P[X_{\pi(1)} \in A_1, \ldots, X_{\pi(n)} \in A_n].\end{aligned}$$

Hence $Q_1 = Q_2$. Therefore,

$$E_P[g(X_1, \ldots, X_n)] = E_{Q_1}[g] = E_{Q_2}[g] = E_P[g(X_{\pi(1)}, \ldots, X_{\pi(n)})].$$

(b) Let π be a permutation such that $\pi(1) = 2,\ \pi(2) = 1,\ \pi(j) = j,\ \forall j \geq 3$. By definition of $\sigma(S)$,

$$\forall A \in \sigma(S),\ \exists B \in \mathcal{B}(\mathbb{R}) \text{ with } A = S^{-1}(B).$$

For $A \in \sigma(S)$,

$$E[X_1 \cdot 1_A] = E[X_1(1_B \circ S)] = E[X_2(1_B \circ S)] = E[X_2 \cdot 1_A],$$

where the second equality follows from part (a) with $g(x_1, \ldots, x_n) = x_1 1_B \circ \left(\sum_{j=1}^{n} x_j\right)$.

Therefore, it holds that

$$E[X_1|S] = E[X_2|S] \quad P\text{-a.s.}$$

Simlarly, $E[X_j|S] = E[X_1|S]$ for $j = 1, \dots, n$, whence

$$S = E[S|S] = E\left[\sum_{j=1}^{n} X_j \middle| S\right] = \sum_{j=1}^{n} E\left[X_j|S\right] = nE[X_1|S] \quad P\text{-a.s.}$$

This implies that

$$E[X_1|S] = \frac{1}{n}S \quad P\text{-a.s.}$$

Solution 8.3 The total number of balls after the n-th iteration is given by $K(n) = s + w + n(t-1)$. For $n \geq 0$, let A_n be the event that the n-th ball to be drawn is black. Then the conditional probability of A_{n+1} given $Y_0, \dots, Y_n$ equals Y_n, that is, for $n \geq 0$,

$$P[A_{n+1}|\sigma(Y_0, \dots, Y_n)] = Y_n. \tag{1}$$

Note that we have, for $n \geq 0$,

$$Y_{n+1}(\omega) = \begin{cases} \frac{Y_n K(n) + (t-1)}{K(n+1)}, & \text{if } \omega \in \ A_{n+1}, \\ \frac{Y_n K(n)}{K(n+1)}, & \text{if } \omega \in \ A_{n+1}^c. \end{cases} \tag{2}$$

Thus we get, setting $\mathcal{F}_n = \sigma(Y_0, \dots, Y_n)$,

$$\begin{aligned} E[Y_{n+1}|\mathcal{F}_n] &= E[Y_{n+1}\mathbf{1}_{A_{n+1}} + Y_{n+1}\mathbf{1}_{A_{n+1}^c}|\mathcal{F}_n] \\ &\overset{(2)}{=} E\left[\frac{Y_n K(n) + (t-1)}{K(n+1)}\mathbf{1}_{A_{n+1}} + \frac{Y_n K(n)}{K(n+1)}\mathbf{1}_{A_{n+1}^c} \middle| \mathcal{F}_n\right] \\ &= \frac{Y_n K(n) + (t-1)}{K(n+1)} P[A_{n+1}|\mathcal{F}_n] + \frac{Y_n K(n)}{K(n+1)} P[A_{n+1}^c|\mathcal{F}_n] \\ &\overset{(1)}{=} \frac{Y_n K(n) + (t-1)}{K(n+1)} Y_n + \frac{Y_n K(n)}{K(n+1)}(1 - Y_n) = Y_n, \end{aligned}$$

since $K(n) + (t-1) = K(n+1)$.