Coordinator
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Probability Theory

Exercise Sheet 9

Exercise 9.1 Let $S, T : \Omega \to \mathbb{N} \cup \{\infty\}$ be $\mathcal{F}_n$-stopping times. Prove or provide a counter example disproving the following statements:

(a) $S - 1$ is a stopping time.

(b) $S + 1$ is a stopping time.

(c) $S \wedge T$ is a stopping time.

(d) $S \vee T$ is a stopping time.

(e) $S + T$ is a stopping time.

Exercise 9.2 Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $(\mathcal{F}_n)_{n\geq 0}$. Let $S \leq T$ be two bounded $(\mathcal{F}_n)_{n\geq 0}$-stopping times and let $(X_n)_{n\geq 0}$ be an $(\mathcal{F}_n)_{n\geq 0}$-submartingale. Show that

$$E[X_T|\mathcal{F}_S] \geq X_S,\ P\text{-a.s.}.$$

(See (3.3.6) on p. 89 of the lecture notes for the definition of $\mathcal{F}_S$.)

Exercise 9.3 Let Y_n, $n \geq 0$ be i.i.d. with $P[Y_0 = 1] = p$ and $P[Y_0 = 0] = 1 - p$ for some $p \in (0, 1)$. Let $\mathcal{F}_n := \sigma(Y_0, \ldots, Y_n)$ for $n \geq 0$ and define

$$T := \inf\{n \geq 0 \mid Y_n = 1\}.$$

Determine the Doob decomposition of $X_n := 1_{\{T\leq n\}}$, $n \geq 0$.
Hint: First check that X_n is an $\mathcal{F}_n$-submartingale. Then try to use Proposition 3.19.

Submission: until 12:00, Nov. 24, through the webpage of the course. You should carefully follow the **submission instructions** on the webpage to get your solutions back.

Office hours: Tue. 15:30-16:30 and Wed. 11:00-12:00 via Zoom with a 10 minutes slot reservation. Organized by the Probability Theory assistants.

Exercise class: Online. Details can be found on the polybox folder of the course.

Exercise sheets and further information are also available on:
https://metaphor.ethz.ch/x/2020/hs/401-3601-00L/

Solution 9.1 First we show that an $\mathcal{F}_n$-stopping time S can be defined equivalently by the conditions $\{S = n\} \in \mathcal{F}_n$ or $\{S \leq n\} \in \mathcal{F}_n$.

If $\{S = n\} \in \mathcal{F}_n$ for all $n \geq 0$, then for all $0 \leq k \leq n$, $\{S = k\} \in \mathcal{F}_k \subseteq \mathcal{F}_n$, which implies that $\{S \leq n\} = \bigcup_{k=0}^{n} \{S = k\} \in \mathcal{F}_n$. On the other hand, $\{S \leq n\} \in \mathcal{F}_n$ for all $n \geq 0$ implies that

- $\{S = 0\} \in \mathcal{F}_0$;
- $\{S \leq n-1\} \in \mathcal{F}_{n-1} \subseteq \mathcal{F}_n$ for all $n \geq 1$. Hence one knows that for all $n \geq 1$, $\{S = n\} = \{S \leq n\} \setminus \{S \geq n-1\} \in \mathcal{F}_n$.

From now on we will use the one most convenient for our purpose in the following.

(a) In general, $S - 1$ need not be a stopping time. Intuitively, this is because to know whether $S - 1$ has 'happened' by time n, information about time $n + 1$ is needed. A counter example can be constructed as follows:

Let $\Omega = \{0, 1\}$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \{\emptyset, \{0\}, \{1\}, \Omega\}$ for $n \geq 1$. We define

$$S = 1_{\{1\}} + 2 \cdot 1_{\{0\}}.$$

Then $\{S \leq 0\} = \emptyset \in \mathcal{F}_0$, $\{S \leq 1\} = \{1\} \in \mathcal{F}_1$, and $\{S \leq k\} = \Omega \in \mathcal{F}_k$ for all $k \geq 2$. Thus, S is a $\mathcal{F}_n$-stopping time. However, $\{S - 1 \leq 0\} = \{S \leq 1\} = \{1\} \notin \mathcal{F}_0$, so $S - 1$ is not a stopping time.

(b) $S + 1$ is a stopping time, since for any $n \geq 0$,

$$\{S + 1 \leq n\} = \{S \leq n - 1\} \in \mathcal{F}_{n-1} \subseteq \mathcal{F}_n.$$

(c) $S \wedge T$ is a stopping time, since for any $n \geq 0$,

$$\{S \wedge T \leq n\} = \{S \leq n\} \cup \{T \leq n\} \in \mathcal{F}_n.$$

(d) $S \vee T$ is a stopping time, since for any $n \geq 0$,

$$\{S \vee T \leq n\} = \{S \leq n\} \cap \{T \leq n\} \in \mathcal{F}_n.$$

(e) $S + T$ is a stopping time, since for any $n \geq 0$,

$$\{S + T = n\} = \bigcup_{k=0}^{n} \underbrace{\{S = k\}}_{\in \mathcal{F}_k \subseteq \mathcal{F}_n} \cap \underbrace{\{T = n - k\}}_{\in \mathcal{F}_{n-k} \subseteq \mathcal{F}_n} \in \mathcal{F}_n.$$

Solution 9.2 Because S, T are bounded, there exists some $k \geq 0$, such that $S \leq T \leq k$ P-almost surely. We then observe that X_S, X_T are integrable because both of them are dominated by the integrable random variable $|X_0| + \ldots + |X_k|$.

Now let $F \in \mathcal{F}_S$. We define a sequence $(C_n)_{n \geq 1}$ of non-negative, bounded random variables through

$$C_n(\omega) := 1_F(\omega) 1_{(S(\omega), T(\omega)]}(n), \quad \omega \in \Omega,\ n \geq 1.$$

Because $\{T \leq n-1\} \in \mathcal{F}_{n-1}$ and $F \cap \{S \leq n-1\} \in \mathcal{F}_{n-1}$, one has that

$$C_n = 1_F 1_{\{S < n\}} 1_{\{T \geq n\}} = 1_{F \cap \{S \leq n-1\}} 1_{\{T \leq n-1\}^c}$$

is $\mathcal{F}_{n-1}$-measurable. This implies that $(C_n)_{n \geq 1}$ is predictable.

By Theorem 3.22, p.93 of the lecture notes, it follows that $C \cdot X$ is a submartingale (with $(C \cdot X)_0 = 0$). Hence it follows that

$$0 \leq E[(C \cdot X)_k] = E\left[\sum_{n=1}^{k} C_n (X_n - X_{n-1})\right] = E\left[(X_T - X_S) 1_F\right].$$

Because $F \in \mathcal{F}_S$ is arbitrary, one has that $E[X_T \mid \mathcal{F}_S] \geq X_S$, P-a.s.

Solution 9.3 As in the hint, we first check that X_n is an $\mathcal{F}_n$-submartingale. Clearly, X_n is $\mathcal{F}_n$-adapted. Furthermore, X_n is bounded for all n, so it is integrable. Finally, $1_{\{T \leq n+1\}} \geq 1_{\{T \leq n\}}$ for every $n \geq 0$, since $\{T \leq n\} \subseteq \{T \leq n+1\}$. Due to this, and by the monotonicity property of conditional expectation, we obtain

$$E[X_{n+1}|\mathcal{F}_n] = E[1_{\{T \leq n+1\}}|\mathcal{F}_n] \geq E[1_{\{T \leq n\}}|\mathcal{F}_n] = 1_{\{T \leq n\}} = X_n \quad P\text{-a.s.}$$

Hence, X_n is an $\mathcal{F}_n$-submartingale, so the Doob decomposition (unique up to P-nullsets) must exist. In other words, there exists a martingale M_n, $n \geq 0$, and a predictable, non-decreasing process A_n, with $A_0 = 0$, such that

$$X_n = M_n + A_n, \quad n \geq 0.$$

To find M_n and A_n, we follow the proof of existence of this decomposition, see Proposition 3.19, p. 90 of the lecture notes. For our X_n, we have for $k \geq 0$:

$$\begin{aligned} E[X_k - X_{k-1}|\mathcal{F}_{k-1}] &= E[1_{\{T \leq k\}} - 1_{\{T \leq k-1\}}|\mathcal{F}_{k-1}] \\ &= E[1_{\{T = k\}}|\mathcal{F}_{k-1}] \\ &= E[1_{\{Y_k = 1\}} 1_{\{T > k-1\}}|\mathcal{F}_{k-1}] \\ &= 1_{\{T > k-1\}} E[1_{\{Y_k = 1\}}|\mathcal{F}_{k-1}] \\ &= 1_{\{T > k-1\}} P[Y_k = 1] \\ &= p 1_{\{T > k-1\}} \; (= A_k - A_{k-1}) \quad P\text{-a.s.,} \end{aligned} \tag{1}$$

since Y is independent of $\mathcal{F}_{k-1}$. Thus, we define

$$A_n := \sum_{k=1}^{n} p 1_{\{T > k-1\}} = p \cdot (T \wedge n), \quad n \geq 0, \tag{2}$$

and

$$M_n := X_n - A_n = 1_{\{T \leq n\}} - p \cdot (T \wedge n), \quad n \geq 0.$$

Therefore, the unique Doob's decomposition of X_n is given by

$$X_n = M_n + A_n = \Big(1_{\{T \leq n\}} - p \cdot (T \wedge n)\Big) + p \cdot (T \wedge n), \quad n \geq 0.$$