Coordinator
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Probability Theory

Exercise Sheet 10

Exercise 10.1 Let $(X_n)_{n\geq 0}$ be a supermartingale with respect to the natural filtration $(\mathcal{F}_n)_{n\geq 0}$. Suppose $X_n \geq 0$ for all $n \geq 0$ and consider the $\mathcal{F}_n$-stopping time $T := \inf\{n \geq 0 : X_n = 0\}$. Show that $X_n = 0$ on the event $\{T < n\}$.

Exercise 10.2 Let $(X_i)_{i\geq 1}$ be i.i.d. random variables with mean 0 and variance $\sigma^2 < \infty$, defined in the probability space $(\Omega, \mathcal{A}, P)$. Set $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ and let T be a $\mathcal{F}_n$-stopping time with $E[T] < \infty$. Define $S_n := \sum_{i=1}^n X_i$ for $n \geq 1$ and $S_0 = 0$.

(a) Show, with the help of the Optional Stopping Theorem, that for all $n \geq 0$,

$$E[S_{T\wedge n}^2] = \sigma^2 E[T \wedge n].$$

(b) Prove that $(S_{T\wedge n})_{n\geq 1}$ is a Cauchy sequence in $L^2(\Omega, \mathcal{A}, P)$.

(c) Show that,

$$E[S_T^2] = \sigma^2 E[T].$$

Exercise 10.3 Consider a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\{\mathcal{F}_n\}_{n\geq 0}$, and let X_n be an $\mathcal{F}_n$-martingale for which $|X_{n+1} - X_n| \leq M$ P-a.s. for some fixed $M < \infty$. Define the events C, D by

$$C := \{\lim X_n \text{ exists and is finite}\},$$
$$D := \{\limsup X_n = +\infty \text{ and } \liminf X_n = -\infty\}.$$

Show that $P[C \cup D] = 1$.
Hint: Show that $P[C^c \cap (\{\sup_{n\in\mathbb{N}} X_n < a\} \cup \{\inf_{n\in\mathbb{N}} X_n > -a\})] = 0$, for all $a > 0$, by considering the processes $\{X_{T_A\wedge n}\}_{n\geq 0}$, for $A = [a, \infty)$ and $A = (-\infty, -a]$, where $T_A = \inf\{n \geq 0 : X_n \in A\}$.

Submission: until 12:00, Dec. 1, through the webpage of the course. You should carefully follow the **submission instructions** on the webpage to get your solutions back.

Office hours: Tue. 15:30-16:30 and Wed. 11:00-12:00 via Zoom with a 10 minutes slot reservation. Organized by the Probability Theory assistants.

Exercise class: Online. Details can be found on the polybox folder of the course.

Exercise sheets and further information are also available on:
https://metaphor.ethz.ch/x/2020/hs/401-3601-00L/

Solution 10.1 Since $X_n \geq 0$,

$$\begin{aligned} 0 \leq E\left[X_n 1_{\{T<n\}}\right] &= \sum_{k=0}^{n-1} E\left[X_n 1_{\{T=k\}}\right] \\ &= \sum_{k=0}^{n-1} E\left[E\left[X_n 1_{\{T=k\}} | \mathcal{F}_k\right]\right] \\ &= \sum_{k=0}^{n-1} E\left[1_{\{T=k\}} E\left[X_n | \mathcal{F}_k\right]\right] \\ &\underset{\text{sup.mart.}}{\leq} \sum_{k=0}^{n-1} E\left[1_{\{T=k\}} X_k\right] = 0. \end{aligned}$$

Therefore, $E\left[X_n 1_{\{T<n\}}\right] = 0$, which implies that $X_n 1_{\{T<n\}} = 0$ P-a.s, since $X_n 1_{\{T<n\}} \geq 0$ P-a.s.

Solution 10.2 Define $M_0 := 0$, $M_n := S_n^2 - n\sigma^2$ for $n \geq 1$, as in Example 3.11 in the lecture notes. Note that S_n and M_n are $\mathcal{F}_n$-martingales (see (3.2.6) and (3.2.8) in the lecture notes).

(a) Due to the Optional Stopping Theorem $M_{n\wedge T}$ is an $\mathcal{F}_n$-martingale, hence

$$0 = E[M_0] = E[M_{0\wedge T}] = E[M_{n\wedge T}] = E[S_{n\wedge T}^2 - (n \wedge T)\sigma^2].$$

Therefore, we conclude that $E[S_{n\wedge T}^2] = \sigma^2 E[n \wedge T]$.

(b) We have that for $n, k \geq 0$,

$$\begin{aligned} ||S_{(n+k)\wedge T} - S_{n\wedge T}||_{L^2}^2 &= E[(S_{(n+k)\wedge T} - S_{n\wedge T})^2] \\ &= E[S_{(n+k)\wedge T}^2] - 2E[S_{(n+k)\wedge T} S_{n\wedge T}] + E[S_{n\wedge T}^2]. \end{aligned} \tag{1}$$

Since S_n is an $\mathcal{F}_n$-martingale we have by the Optional Stopping Theorem that $S_{n\wedge T}$ is as well a $\mathcal{F}_n$-martingale. Therefore, we get for the second term in the last equality that,

$$\begin{aligned} E[S_{(n+k)\wedge T} S_{n\wedge T}] &= E[E[S_{(n+k)\wedge T} S_{n\wedge T} | \mathcal{F}_n]] \\ &= E[S_{n\wedge T} E[S_{(n+k)\wedge T} | \mathcal{F}_n]] \\ &= E[S_{n\wedge T}^2] \end{aligned}$$

Plugging this in (1) and using part (a), we get

$$\begin{aligned} & ||S_{(n+k)\wedge T} - S_{n\wedge T}||_{L^2}^2 = E[S_{(n+k)\wedge T}^2] - E[S_{n\wedge T}^2] = \sigma^2(E[T \wedge (n+k)] - E[T \wedge n]) \\ = \;& \sigma^2(E[(T \wedge (n+k) - T \wedge n)1_{\{n<T\}}] + \underbrace{E[(T \wedge (n+k) - T \wedge n)1_{\{n \geq T\}}]}_{=0}) \\ = \;& \sigma^2 E[(T \wedge (n+k) - T \wedge n)1_{\{n \leq T\}}] \\ \leq \;& \sigma^2 E[T \cdot 1_{\{n<T\}}] \to 0. \end{aligned}$$

The last convergence follows from dominated convergence and the fact that $E[T] < \infty$.

(c) Since $L^2(\Omega, \mathcal{A}, P)$ is a complete space, every Cauchy sequence converges. In particular, by part (b) we have that $S_{n\wedge T} \xrightarrow{L^2} S_T$, which implies $E[S_{n\wedge T}^2] \to E[S_T^2]$. Then

$$E[S_T^2] = \lim_{n\to\infty} E[S_{n\wedge T}^2] \overset{\text{(a)}}{=} \sigma^2 \lim_{n\to\infty} E[T\wedge n] \overset{\text{mon. conv.}}{=} \sigma^2 E[T].$$

Solution 10.3 Without loss of generality, we assume that $X_0 = 0$ or we just replace X_n by $X_n - X_0$.

Note that the hitting time T_A is an $\{\mathcal{F}_n\}_{n\geq 0}$-stopping time, for any $A \in \mathcal{B}(\mathbb{R})$, as (3.3.3) in Example 3.17, p. 89 of the lecture notes. Thus, from the optional stopping theorem ((3.4.15), p. 93 of the lecture notes), $X_{T_A\wedge n}$ is an $\{\mathcal{F}_n\}_{n\geq 0}$-martingale. If we let $A = [a, \infty)$ for $a > 0$, we furthermore have that

$$X_{T_{[a,\infty)}\wedge n} \leq a + M,$$

because of the bounded increments of X_n and $X_0 = 0$. This implies that we have

$$\sup_{n\geq 0} E\left[\left(X_{T_{[a,\infty)}\wedge n}\right)^+\right] \leq a + M < \infty.$$

Thus, by the martingale convergence theorem, (3.4.23), p. 96 of the lecture notes, the martingale $X_{T_{[a,\infty)}\wedge n}$ converges to some integrable random variable. But on the event $\{\sup_{n\geq 0} X_n < a\}$, we have $T_{[a,\infty)} = \infty$, so that $X_{T_{[a,\infty)}\wedge n} = X_n$ for all n. Thus on this event, X_n converges to a finite limit. From the definition of C, we obtain

$$P\left[C^c \cap \left\{\sup_{n\geq 0} X_n < a\right\}\right] = 0, \tag{2}$$

for all $a > 0$. Similarly by considering $-X_{T_{(-\infty,-a]}}$, or by symmetry, we can obtain that for all $a > 0$

$$P\left[C^c \cap \left\{\inf_{n\geq 0} X_n > -a\right\}\right] = 0. \tag{3}$$

Now by equations (2) and (3), we have

$$P\left[C^c \cap \left(\left\{\sup_{n\geq 0} X_n < a\right\} \cup \left\{\inf_{n\geq 0} X_n > -a\right\}\right)\right] = 0. \tag{4}$$

Taking the limit $a \to \infty$, and using the continuity property of measures, we get by definition of the event D

$$P[C^c \cap D^c] = 0. \tag{5}$$

Now the claim follows by taking the complement event in (5).

Remark: This exercise is the essential ingredient of the proof of the generalised version of the second Borel-Cantelli Lemma, see Theorems 5.31 and 5.32 in Durrett's book (pp. 204-205 in 4th online edition, pp. 239-240 in 3rd edition).