Probability Theory

Exercise Sheet 11

Exercise 11.1 (The generalized Borel-Cantelli lemma)
Consider $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}_n\}_{n\geq 0}$, and let $A_n \in \mathcal{F}_n$, $n \geq 1$, be a sequence of events. Show that, up to a P-nullset,

$$\limsup_{n\to\infty} A_n = \{\sum_{n\geq 1} P[A_n|\mathcal{F}_{n-1}] = \infty\}.$$

Hint: Use Exercise 10.3.

Exercise 11.2 Consider $Y, X_i, i \geq 1$, independent random variables with $Y \geq 0$, integer valued such that $E[Y] = \mu \in (1, \infty)$, and $X_i, i \geq 1$, i.i.d. Bernoulli random variables with $P[X_i = 0] = q \in (0, 1)$. If $S_m, m \geq 0$, denotes the partial sums of the X_i, let ν be the law of S_Y. Consider the Galton-Watson chain $Z_n, n \geq 0$ with offspring distribution ν (see p. 97 of the Lecture Notes).

(a) For which values of q is the Galton-Watson chain subcritical?

(b) If Y is constant and equal to 2, find

$$f(q) := P[Z_n > 0, \text{ for all } n \geq 0].$$

Hint: See Lecture Notes p. 100.

Exercise 11.3 Let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of independent, non-negative random variables with expectation 1. Consider the natural filtration $(\mathcal{F}_n)_{n\geq 0}$. We define

$$M_0 = 1, \quad M_n = Y_1 Y_2 \cdots Y_n, \text{ for } n \in \mathbb{N}.$$

(a) Prove that $(M_n)_{n\in\mathbb{N}}$ is a non-negative martingale with respect to the filtration $(\mathcal{F}_n)_{n\geq 0}$ and there exists a random variable M_∞, so that $M_n \to M_\infty$ a.s.

(b) Let $a_n := E[\sqrt{Y_n}]$. Show that $a_n \in (0, 1]$.

(c) Show that if $\prod_n a_n > 0$, it holds that $M_n \to M_\infty$ in L^1 and $E[M_\infty] = 1$.
Hint: Let $\hat{Y}_n := \sqrt{Y_n}/a_n$ and $\hat{M}_n := \hat{Y}_1\hat{Y}_2, \ldots, \hat{Y}_n$ for $n \geq 1$, $\hat{M}_0 = 1$. Note that $M_n \leq \hat{M}_n^2$ for $n \in \mathbb{N}$. Then use (a) together with Doob's inequality to conclude the proof.

(d) Show that if $\prod_n a_n = 0$, then $M_\infty = 0$ a.s.

Submission: until 12:00, Dec. 8, through the webpage of the course. You should carefully follow the **submission instructions** on the webpage to get your solutions back.

Office hours: Tue. 15:30-16:30 and Wed. 11:00-12:00 via Zoom with a 10 minutes slot reservation. Organized by the Probability Theory assistants.

Exercise class: Online. Details can be found on the polybox folder of the course.

Exercise sheets and further information are also available on:
https://metaphor.ethz.ch/x/2020/hs/401-3601-00L/

Solution 11.1 We define $X_0 := 0$, $X_n := \sum_{m=1}^n \big(1_{A_m} - P[A_m|\mathcal{F}_{m-1}]\big)$, $n \geq 1$. Then X_n is an $\mathcal{F}_n$-martingale, since

$$E[X_{n+1} - X_n|\mathcal{F}_n] = E\left[1_{A_{n+1}} - P[A_{n+1}|\mathcal{F}_n]\middle|\mathcal{F}_n\right] = 0.$$

Furthermore $|X_{n+1} - X_n| \leq 2$. We apply the result of Exercise 10.3 to obtain $P[C \cup D] = 1$. Note that:

- $\sum_{n\geq 1} 1_{A_n} = \infty \Longleftrightarrow \sum_{n\geq 1} P[A_n|\mathcal{F}_{n-1}] = \infty$ on C.
- $\sum_{n\geq 1} 1_{A_n} = \infty$ and $\sum_{n\geq 1} P[A_n|\mathcal{F}_{n-1}] = \infty$ on D.

Using that $P[C \cup D] = 1$, we get that for an event N with $P[N] = 0$,

$$\left\{\sum_{n\geq 1} 1_{A_n} = \infty\right\} \cap N^c = \left\{\sum_{n\geq 1} P[A_n|\mathcal{F}_{n-1}] = \infty\right\} \cap N^c.$$

Finally, the claim follows since

$$\limsup_{n\to\infty} A_n = \left\{\sum_{n\geq 1} 1_{A_n} = \infty\right\}.$$

Solution 11.2

(a) Let us observe that

$$E[S_Y] = E\left[\sum_{n=0}^{\infty} S_n 1_{\{Y=n\}}\right] \overset{\text{MCT+ind.}}{=} \sum_{n=0}^{\infty} E[S_n] P[Y = n]$$
$$\overset{\text{i.i.d.}}{=} \sum_{n=1}^{\infty} E[X_1] \cdot n P[Y = n] = (1-q)E[Y] = (1-q)\mu.$$

We know that this Galton-Watson process is subcritical if and only if $E[S_Y] < 1$, that is, when $q > 1 - 1/\mu$.

(b) Let us observe that when $Y = 2$, we have

$$m = E[S_Y] = 2(1-q) \begin{cases} < 1 & \text{if } q \in (\frac{1}{2}, 1], \\ = 1 & \text{if } q = \frac{1}{2}, \\ > 1 & \text{if } q \in [0, \frac{1}{2}). \end{cases}$$

Thus, if $q \in (\frac{1}{2}, 1]$, our Galton-Watson process is subcritical, if $q = \frac{1}{2}$ it is critical, and if $q \in [0, \frac{1}{2})$ it is supercritical.

For a subcritical Galton-Watson process, we have $P[Z_n = 0 \text{ eventually}] = 1$ by (3.5.7), p. 99, and by (3.5.10), p. 100 of the lecture notes also for a critical process. Hence,

$$f(q) = 0 \quad \forall q \in [1/2, 1].$$

In the supercritical case, we have, by (3.5.13), p. 101 of the lecture notes,

$$P[Z_n = 0 \text{ eventually}] = \varrho \in [0, 1),$$

where ϱ is the unique solution to $\varrho = \varphi(\varrho)$ in $[0, 1)$, and let X be a random variable with distribution ν, we have

$$\varphi(z) = E[z^X] = \sum_{k=0}^{2} P[X = k] z^k = q^2 + 2q(1-q)z + (1-q)^2 z^2,$$

(see (3.5.11), p. 100 of the lecture notes, and the explanations right below it). Solving the quadratic equation

$$\varphi(z) = \big(q + (1-q)z\big)^2 = z$$

for z, we obtain the solutions $z = 1$ and $z = \frac{q^2}{(1-q)^2}$. Thus, the unique solution to $\varphi(\varrho) = \varrho$ in $[0, 1)$ is

$$\varrho = \frac{q^2}{(1-q)^2},$$

from which it follows that

$$f(q) = 1 - P[Z_n = 0 \text{ eventually}] = 1 - \varrho = 1 - \frac{q^2}{(1-q)^2} = \frac{1-2q}{(1-q)^2},$$

for $q \in [0, 1/2)$.

Comment: ν has the following interpretation. Given an infected individual having a random number of contact individuals with the same distribution as Y. Suppose each contact individual is independently susceptible with probability $p = 1 - q$ and immune with probability q. Then ν describes the law of infected contacts of the original (infected) individual.

Solution 11.3

(a) Since $(M_n)_{n\geq 0}$ is adapted to the filtration $\mathcal{F}_n = \sigma(Y_0, \dots, Y_n)_{n\in\mathbb{N}}$ and, for $n \in \mathbb{N}$, $E[|M_n|] = E[Y_1] \cdots E[Y_n] = 1$, it holds that $M_n \in L^1$. Moreover,

$$\begin{aligned} E[M_{n+1}|\mathcal{F}_n] &= E[M_n Y_{n+1}|\mathcal{F}_n] \\ {\scriptstyle (M_n\ \mathcal{F}_n\text{-meas.})} &= M_n E[Y_{n+1}|\mathcal{F}_n] \\ {\scriptstyle (Y_{n+1},\ \mathcal{F}_n\ \text{indep.})} &= M_n E[Y_{n+1}] = M_n. \end{aligned}$$

Hence, $(M_n)_{n\geq 0}$ is a $\mathcal{F}_n$-martingale with $\sup_{n\in\mathbb{N}} E[M_n] = 1 < \infty$ and, by the martingale convergence theorem ((3.4.23) in the lecture notes), M_n converges to an integrable random variable M_∞ P-a.s.

(b) From the Cauchy-Schwarz (or Hölder) inequality it follows that,

$$a_n = E[\sqrt{Y_n}] \leq E[Y_n]^{1/2} E[1]^{1/2} = 1.$$

Thus, since $Y_n \geq 0$ P-a.s. and therefore $\sqrt{Y_n} \geq 0$ P-a.s. and $E[Y_n] = 1$, it must hold that $a_n > 0$.

(c) We define $\hat{Y}_n = \sqrt{Y_n}/a_n$ and $\hat{M}_n = \hat{Y}_1 \hat{Y}_2, \dots, \hat{Y}_n$, $\hat{M}_0 = 1$. Then, from (a), the process $(\hat{M}_n)_{n\in\mathbb{N}}$ is also a non-negative martingale, converging to an integrable random variable $\hat{M}_\infty$. Furthermore $M_n \leq \hat{M}_n^2$ for $n \in \mathbb{N}$.

Suppose that $\prod_n a_n > 0$. Then we get,

$$E[\hat{M}_n^2] = (a_1 a_2 \cdots a_n)^{-2} \leq \left(\prod_n a_n\right)^{-2} < \infty.$$

Therefore, $\hat{M}_n \in L^2$, and by Doob's inequality (see Corollary 3.34),

$$E\left[\sup_{n\in\mathbb{N}} M_n\right] \leq E\left[\sup_{n\in\mathbb{N}} \hat{M}_n^2\right] \leq 4 \sup_{n\in\mathbb{N}} E[\hat{M}_n^2] < \infty.$$

From Lebesgue convergence theorem ($|M_n - M_\infty| \leq 2\sup_{n\in\mathbb{N}} M_n \in L^1$) we obtain convergence also in L^1.

(d) Suppose that $\prod_n a_n = 0$. Then, from the P-a.s. convergence of $\hat{M}_n = \sqrt{M_n}/(\prod_{i=1}^n a_i)$ to an a.s. finite random variable $\hat{M}_\infty \in L^1$, it follows that $M_\infty = 0$ a.s.