ETH Zurich HS 2020
D-MATH, D-PHYS
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Probability Theory

Exercise Sheet 12

Exercise 12.1 Let X_n, $n \geq 0$, be a uniformly integrable submartingale and N a stopping time.

(a) Show that $\sup_n E[X^+_{N\wedge n}] \leq \sup_n E[X^+_n] < \infty$.

(b) Show that X_N (where $X_N 1_{\{N=\infty\}} = 1_{\{N=\infty\}} \lim_n X_n$) is integrable.

(c) Show that $X_{N\wedge n}$, $n \geq 0$, is a uniformly integrable submartingale.

(d) Show that $X_{N\wedge n}$ converges P-a.s. and in L^1 to X_N.

Exercise 12.2 Let $(X_n)_{n\geq 0}$ be a uniformly integrable family of random variables on $(\Omega, \mathcal{A}, P)$.

(a) Assume that X_n converges to a random variable X in distribution. Show that

$$E[X_n] \xrightarrow{n\to\infty} E[X].$$

Remark: Compare to (3.6.18)–(3.6.20), p. 112 of the lecture notes.

(b) Assume that X_n converges to a random variable X in probability. Show that $X \in L^1$ and that X_n converges to X in L^1.

Exercise 12.3 *Azuma's inequality.* Let $0 = X_0, \dots, X_m$ be a martingale with $|X_{i+1} - X_i| \leq 1$ for all $0 \leq i < m$. Let $\lambda > 0$ be arbitrary.

(a) Show that $E[e^{\alpha(X_i - X_{i-1})} | X_{i-1}, X_{i-2}, \dots, X_0] \overset{(1)}{\leq} \cosh\alpha \overset{(2)}{\leq} e^{\alpha^2/2}$.

Hint: For (1) use that for $y \in [-1,1]$, $e^{\lambda y} \leq \frac{e^\lambda + e^{-\lambda}}{2} + y\frac{e^\lambda - e^{-\lambda}}{2}$. Inequality (2) follows from the series expansion of $\cosh\alpha$.

(b) Show that $E[e^{\alpha X_m}] \leq e^{\alpha^2 m/2}$.

(c) Show that $P\left[X_m > \lambda\sqrt{m}\right] < e^{-\lambda^2/2}$.

Submission: until 12:00, Dec. 15, through the webpage of the course. You should carefully follow the **submission instructions** on the webpage to get your solutions back.

Office hours: Tue. 15:30-16:30 and Wed. 11:00-12:00 via Zoom with a 10 minutes slot reservation. Organized by the Probability Theory assistants.

Exercise class: Online. Details can be found on the polybox folder of the course.

Exercise sheets and further information are also available on:
https://metaphor.ethz.ch/x/2020/hs/401-3601-00L/

Solution 12.1

(a) Define $H_k = 1_{\{N<k\}}$. Since H is predictable, $(H \cdot X)_n = X_n - X_{N\wedge n}$ is a submartingale. Hence, for any $n \geq 0$,

$$0 = E[(H \cdot X)_0] \leq E[(H \cdot X)_n] = E[X_n] - E[X_{N\wedge n}].$$

This holds for any submartingale. Since X^+ is also a submartingale and n is arbitrary,

$$\sup_n E[X^+_{N\wedge n}] \leq \sup_n E[X^+_n] < \infty,$$

where the last inequality follows from the uniform integrability.

(b) From (a) and the martingale convergence theorem, $X_{N\wedge n} \to X_N$ P-a.s. and $E[|X_N|] < \infty$.

(c) $X_{N\wedge n}$ is submartingale follows from the optional stopping theorem. Finally,

$$\begin{aligned} E\left[|X_{N\wedge n}|1_{\{|X_{N\wedge n}|>K\}}\right] &= E\left[|X_N|1_{\{|X_N|>K\}}1_{\{N\leq n\}}\right] + E\left[|X_n|1_{\{|X_n|>K\}}1_{\{N>n\}}\right] \\ &\leq E\left[|X_N|1_{\{|X_N|>K\}}\right] + E\left[|X_n|1_{\{|X_n|>K\}}\right], \end{aligned}$$

so the uniform integrability follows directly from the uniform integrability of X.

(d) From the solution of part (b) we have that $X_{N\wedge n}$ converges P-a.s. to X_N. From (c) we also know that $X_{N\wedge n}$ is uniformly integrable. Then, using Proposition 3.41 on p. 111 of the Lecture Notes, we conclude that $X_{N\wedge n}$ converges to X_N in L_1.

Solution 12.2

(a) Since $X_n \xrightarrow{n\to\infty} X$ in distribution, we know by Proposition 2.7, p. 50 of the lecture notes that one can construct random variables Y_n, $n \geq 0$, and Y on a common probability space $(\Omega', \mathcal{A}', P')$, such that $Y_n \overset{d}{=} X_n$, for all $n \geq 0$, $Y \overset{d}{=} X$, and $Y_n \to Y$, P'-almost surely. It is easy to verify that the family $\{Y_n\}_{n\geq 0}$ is also uniformly integrable, since

$$\lim_{M\to\infty} \sup_{n\geq 0} E_{P'}\left[|Y_n|1_{\{|Y_n|>M\}}\right] \overset{Y_n \overset{d}{=} X_n}{=} \lim_{M\to\infty} \sup_{n\geq 0} E_P\left[|X_n|1_{\{|X_n|>M\}}\right] = 0.$$

Using Fatou's Lemma, we obtain

$$\begin{aligned} E_P[|X|] = E_{P'}[|Y|] &\overset{\text{Fatou}}{\leq} \liminf_{n\to\infty} E_{P'}[|Y_n|] \leq M + \sup_{n\geq 0} E_{P'}\left[|Y_n|1_{\{|Y_n|\geq M\}}\right] \\ &\leq M + \epsilon < +\infty. \end{aligned} \tag{1}$$

Therefore $Y \in L^1(\Omega', \mathcal{A}', P')$. So by (3.6.18)-(3.6.19), p. 112 of the lecture notes, we have

$$E_{P'}[Y_n] \xrightarrow{n\to\infty} E_{P'}[Y],$$

and the result follows since $E_P[X_n] = E_{P'}[Y_n]$, for all $n \geq 0$, and $E_P[X] = E_{P'}[Y]$.

(b) Since convergence in probability implies convergence in distribution, we can construct Y_n, for $n \geq 1$, and Y defined on $(\Omega', \mathcal{A}', P')$ such that Y_n converges to $Y \in L^1(\Omega', \mathcal{A}', P')$ P'-almost surely, as in (a). Consider an arbitrary $\epsilon > 0$. Since the family $\{Y_n\}_{n\geq 0}$ is uniformly integrable (see (a) again), there exists $M \in \mathbb{R}$ such that $\sup_{n\geq 0} E_{P'}[|Y_n|1_{\{|Y_n|>M\}}] \leq \epsilon$. Analogously to

(1) in (a), Fatou's lemma gives $E_P[|X|1_{\{|X|>M\}}] \le \epsilon$. Moreover, by convergence in probability, there exists $n_0 \ge 0$ such that, for all $n \ge n_0$, we have

$$P\Big[\underbrace{|X_n - X| \ge \epsilon}_{A_n}\Big] < \frac{\epsilon}{M}.$$

Hence, for all $n \ge n_0$, we obtain

$$\begin{aligned}
E_P[|X_n - X|] &\le E_P\left[|X_n - X|1_{\{|X_n|\le M, |X|\le M\}}\right] \\
&+ E_P\left[|X_n - X|1_{\{|X_n|>M\}}1_{\{|X_n|\ge|X|\}}\right] + E_P\left[|X_n - X|1_{\{|X|>M\}}1_{\{|X|\ge|X_n|\}}\right] \\
&\le E_P\left[|X_n - X|1_{\{|X_n|\le M, |X|\le M\}}\right] \\
&+ 2\underbrace{E_P\left[|X_n|1_{\{|X_n|>M\}}\right]}_{\le\epsilon} + 2\underbrace{E_P\left[|X|1_{\{|X|>M\}}\right]}_{\le\epsilon} \\
&\le E_P\Big[\underbrace{|X_n - X|1_{\{|X_n|\le M, |X|\le M\}}}_{\le 2M}1_{A_n}\Big] \\
&+ E_P\Big[\underbrace{|X_n - X|1_{\{|X_n|\le M, |X|\le M\}}}_{\le\epsilon}1_{A_n^c}\Big] + 4\epsilon \\
&\le 2MP[A_n] + 5\epsilon \le 7\epsilon.
\end{aligned}$$

Therefore, X_n converges to X in L^1.

Solution 12.3

(a) Let $Y_i = X_i - X_{i-1}$ with $|Y_i| \le 1$ and $E[Y_i|X_{i-1}, X_{i-2}, \dots, X_0] = 0$. We can then show that

$$E[e^{\alpha X_i}|X_{i-1}, X_{i-2}, \dots, X_0] \overset{(1)}{\le} \cosh\alpha \overset{(2)}{\le} e^{\alpha^2/2}.$$

(1): For any $y \in [-1, 1]$,

$$e^{\lambda y} \le \frac{e^{\lambda} + e^{-\lambda}}{2} + y\frac{e^{\lambda} - e^{-\lambda}}{2}.$$

(2): We expand $\cosh\alpha$ to obtain

$$\cosh\alpha = \sum_{k\ge0}\frac{\alpha^{2k}}{(2k)!} \le \sum_{k\ge0}\frac{\alpha^{2k}}{2^k k!} = e^{\alpha^2/2}.$$

(b) It follows from (a) that

$$\begin{aligned}
E\left[e^{\alpha X_m}\right] &= E\left[\prod_{i=1}^{m} e^{\alpha Y_i}\right] \\
&= E\left[\left(\prod_{i=1}^{m-1} e^{\alpha Y_i}\right) E\left[e^{\alpha Y_m}|X_{m-1}, X_{m-2}, \dots, X_0\right]\right] \\
&\le E\left[\prod_{i=1}^{m-1} e^{\alpha Y_i}\right] e^{\alpha^2/2} \le e^{\alpha^2 m/2}.
\end{aligned}$$

(c) With $\alpha = \frac{\lambda}{\sqrt{m}}$, it follows from (b) that

$$\begin{aligned} P[X_m > \lambda\sqrt{m}] &= P\left[e^{\alpha X_m} > e^{\alpha\lambda\sqrt{m}}\right] \\ &< E\left[e^{\alpha X_m}\right] e^{-\alpha\lambda\sqrt{m}} \\ &\leq e^{\alpha^2 m/2 - \alpha\lambda\sqrt{m}} = e^{-\lambda^2/2}. \end{aligned}$$