Coordinator
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Probability Theory

Exercise Sheet 13

Definition: Let $(\Omega, \mathcal{F}, (P_x)_{x\in E})$ be a canonical (time-homogenous) Markov chain with a *countable* state space E, a transition kernel K, and canonical coordinates $(X_n)_{n\geq 0}$. The matrix

$$Q = (Q(x,y))_{x,y\in E} := (K(x,\{y\}))_{x,y\in E} = (P_x[X_1 = y])_{x,y\in E}$$

is then called the *transition matrix* of the Markov chain. For the meanings of notation P_x and transition kernel we refer to p. 145 in lecture notes.

Exercise 13.1 Let $(\Omega, \mathcal{F}, (P_x)_{x\in E})$ be a canonical time–homogeneous Markov chain with a countable state space E, canonical coordinate process $(X_n)_{n\geq 0}$ and transition kernel K. Let $A \subset E$ and τ_A the first entrance time of A, i.e., $\tau_A := \inf\{n \geq 0 \,|\, X_n \in A\}$. Suppose that there exists $n \geq 1$ and $\alpha > 0$ such that for all $x \in A^c$,

$$P_x[X_n \in A] = \sum_{a\in A} P_x[X_n = a] \geq \alpha.$$

Show that for all $x \in E$ we have that $P_x(\tau_A < +\infty) = 1$.

Exercise 13.2 Let E be a countable set, $(S, \mathcal{S})$ a measurable space, $(Y_n)_{n\geq 1}$ a sequence of i.i.d. S-valued random variables. We define a sequence $(Z_n)_{n\geq 0}$ through $Z_0 = x \in E$ and $Z_{n+1} = \Phi(Z_n, Y_{n+1})$, where $\Phi : E \times S \to E$ is a measurable map. Find a transition kernel K on E such that the canonical law P_x with transition kernel K has the same law as $(Z_n)_{n\geq 0}$ (hence $(Z_n)_{n\geq 0}$ induces a time-homogenous Markov chain with transition kernel K). Calculate the corresponding transition matrix.

Exercise 13.3 *(Probabilistic solution to the Dirichlet problem).*
Consider $(X_n)_{n\geq 0}$ the canonical Markov chain on $\mathbb{Z}^d$ with transition kernel

$$K(x, dy) = \frac{1}{2d} \sum_{e\in\mathbb{Z}^d:|e|=1} \delta_{x+e}(dy),$$

corresponding to the simple random walk on $\mathbb{Z}^d$. Let $U \neq \emptyset$ be a finite subset of $\mathbb{Z}^d$.

(a) If $T_U = \inf\{n \geq 0; X_n \notin U\}$ stands for the exit time of U, show that for all $x \in \mathbb{Z}^d$, P_x-a.s., $T_U < \infty$.

Hint: Show that $M_n = \sum_{1\leq i\leq d} X_n \cdot e_i$, $n \geq 0$ (with $e_1, \dots, e_d$ the canonical basis of $\mathbb{Z}^d$) is a martingale with bounded increments and use Exercise 10.3.

(b) Let g be a bounded function on $\mathbb{Z}^d \setminus U$. If $f : \mathbb{Z}^d \to \mathbb{R}$ solves the Dirichlet problem

$$(*) \begin{cases} \frac{1}{2d} \sum_{y:|y-x|=1} f(y) = f(x), & \text{for } x \in U, \\ f(x) = g(x), & \text{for } x \notin U. \end{cases}$$

Show that necessarily $f(x) = E_x[g(X_{T_U})]$ for all $x \in \mathbb{Z}^d$.

Hint: Use the martingale (4.2.58) in the lecture notes and the Optional Stopping Theorem.

(c) Show, without using (b), that the function $f(x) = E_x[g(X_{T_U})]$, $x \in \mathbb{Z}^d$ solves $(*)$.

Hint: distinguish the cases $x \notin U$ and $x \in U$. When $x \in U$ note that P_x-a.s., $g(X_{T_U}) = g(X_{T_U}) \circ \theta_1$ and use the Markov property (4.2.55).

Submission: until 12:00, Dec. 22, through the webpage of the course. You should carefully follow the **submission instructions** on the webpage to get your solutions back.

Office hours: Tue. 15:30-16:30 and Wed. 11:00-12:00 via Zoom with a 10 minutes slot reservation. Organized by the Probability Theory assistants.

Exercise class: Online. Details can be found on the polybox folder of the course.

Exercise sheets and further information are also available on:
https://metaphor.ethz.ch/x/2020/hs/401-3601-00L/

Solution 13.1 Let $x \in A$, $\tau_A = 0$ P_x-a.s. For $x \in A^c$, we have that for all $k \geq 0$,

$$P_x(\tau_A > (k+1)n) \leq P_x(\tau_A > kn, X_{(k+1)n} \in A^c) = E_x[E_x[1_{\{\tau_A > kn\}} 1_{\{X_{(k+1)n} \in A^c\}} | \mathcal{F}_{nk}]]$$
$$\overset{\text{Markov}}{=} E_x[1_{\{\tau_A > kn\}} \underbrace{P_{X_{kn}}[X_n \in A^c]}_{\leq 1-\alpha}] \leq (1-\alpha) P_x(\tau_A > kn).$$

From the last we get by induction that $P_x(\tau_A > kn) \leq (1-\alpha)^k$ and taking the limit as k goes to infinity we get that,

$$P_x(\tau_A = +\infty) = \lim_{k\to\infty} P_x(\tau_A > kn) = 0.$$

Solution 13.2 Define for each $x, y \in E$, $K(x, \{y\}) := P[\Phi(x, Y_1) = y]$. Consider the probability measure $P_x = P_{\delta_x}$ on $E^{\mathbb{N}}$ as in (4.2.53) on p. 144 in lecture notes with transitional kernel K and initial distribution $\mu := \delta_x$, where δ denotes the Dirac-delta function. (Note that the existence of P_x is provided by Ionescu-Tulcea theorem). We need to show that P_x has the same law as $(Z_n)_{n\geq 0}$.

It is sufficient to show that for any $n \geq 0$ and bounded functions $f_0, f_1, \ldots, f_n : E \to \mathbb{R}$,

$$E[f_0(Z_0)f_1(Z_1)\ldots f_n(Z_n)] = E^{P_x}[f_0(X_0)f_1(X_1)\ldots f_n(X_n)], \tag{1}$$

where $(X_n)_{n\geq 0}$ is the canonical coordinate process on E, i.e. for each $n \geq 0$ and $e = (e_1, e_2, \ldots, e_n)$, $X_n(e) = e_n$. We are going to use induction. For $n = 0$, $Z_0 = x$ and $X_0 = x$ P_x-a.s. (cf. (4.2.53)), hence $E[f_0(Z_0)] = f_0(x) = E[f_0(X_0)]$. For the induction step, fix $n > 1$ and assume that (1) holds for n. For any $f : E \to \mathbb{R}$, define $Kf(x) := \sum_{e\in E} K(x, \{e\})f(e)$ and note that by the i.i.d. property of $(Y_n)_{n\geq 1}$, for each $n \geq 0$ we have

$$\begin{aligned}
E\left[f\left(\Phi(Z_{n+1})\right)|\sigma(Z_0,\ldots,Z_n)\right] &= E\left[f\left(\Phi(Z_n, Y_{n+1})\right)|\sigma(Z_0,\ldots,Z_n)\right] \\
&= \sum_{z\in E}\sum_{e\in E} P[\Phi(z, Y_{n+1}) = e]f(e)1_{\{Z_n = z\}} \\
&= \sum_{z\in E} Kf(z)1_{\{Z_n=z\}} = Kf(Z_n).
\end{aligned}$$

Hence with $f_n' := f_n K f_{n+1}$ it follows for the LHS of (1) that

$$\begin{aligned}
E[f_0(Z_0)f_1(Z_1)\ldots f_{n+1}(Z_{n+1})] &= E\left[f_0(Z_0)f_1(Z_1)\ldots f_n(Z_n)E\left[f_{n+1}(Z_{n+1})|\sigma(Z_0,\ldots,Z_n)\right]\right] \\
&= E\left[f_0(Z_0)f_1(Z_1)\ldots f_n(Z_n)Kf_{n+1}(Z_n)\right] \\
&= E\left[f_0(Z_0)f_1(Z_1)\ldots f_{n-1}(Z_{n-1})f_n'(Z_n)\right].
\end{aligned}$$

For the RHS of (1), we obtain by (4.2.53)

$$\begin{aligned}
E^{P_x}[f_0(X_0)f_1(X_1)\ldots f_{n+1}(X_{n+1})] &= E^{P_x}\left[f_0(X_0)f_1(X_1)\ldots f_n(X_n)Kf_{n+1}(X_n)\right] \\
&= E^{P_x}\left[f_0(X_0)f_1(X_1)\ldots f_{n-1}(X_{n-1})f_n'(X_n)\right],
\end{aligned}$$

and hence (1) for $n+1$ follows from the induction hypothesis.

This shows that $(Z_n)_{n\geq 0}$ is a time homogeneous Markov chain and the transition matrix is given through

$$Q(x,y) = P_x[X_1 = y] = E^{P_x}\left[E[1_{\{X_1=y\}}|\mathcal{F}_0]\right] = P[\Phi(x, Y_1) = y].$$

We remark that the time homogeneity of $(X_n)_{n\geq 0}$ follows from the observation that for all $n \geq 0$ and $x, y \in E$,

$$P[X_{n+1} = y | X_n = x] = P[X_1 = y | X_0 = x] = Q(x,y).$$

Solution 13.3

(a) Notice that M_n is $\mathcal{F}_n$-adapted, where $F_n = \sigma(X_0, \ldots, X_n)$, $n \geq 0$. Also, since for all x, X_n is P_x-integrable we have that M_n also is. By the definition of the canonical Markov chain (4.2.53) and the Markov property in the form (4.2.55), we have that for $n \geq 0$

$$\begin{aligned}
E_x[M_{n+1}|\mathcal{F}_n] &= E^{P_{X_n}}\left[\sum_{1\leq i\leq d} X_1 \cdot e_i\right] \\
&= \sum_{1\leq i\leq d}\left(\frac{1}{2d}\sum_{e\in\mathbb{Z}^d:|e|=1}(X_n+e)\cdot e_i\right) \\
&= \sum_{1\leq i\leq d}\left(X_n\cdot e_i + \underbrace{\tfrac{1}{2d}\sum_{e\in\mathbb{Z}^d:|e|=1} e_i}_{=0}\right) = M_n
\end{aligned}$$

which means that M_n is a martingale. It is clear that for all $n \geq 0$, $|M_n - M_{n+1}| = 1$, and by Exercise 10.3 we have that P_x-a.s. $(M_n)_{n\geq 0}$ is either converging to a finite limit, or it visits infinitely often $+\infty$ and $-\infty$. Since $|M_n - M_{n+1}| = 1$, we have that the limit cannot be finite. Therefore, P_x-a.s. $\limsup M_n = +\infty$ or $\liminf M_n = -\infty$. Since U is finite, this implies that $T_U < \infty$ P_x-a.s.

(b) Let f be a solution of $(*)$. Since U is finite and g is bounded, we have that f is bounded. Let $x \in \mathbb{Z}^d$ and $\mu = \delta_x$. By Proposition 4.34. of the lecture notes, the process

$$\begin{aligned}
M_n &= f(X_n) - \sum_{k=0}^{n-1}(Kf - f)(X_k), \quad n \geq 1 \\
M_0 &= f(X_0)
\end{aligned}$$

is a martingale with respect to to the measure $P_\mu = P_x$. By the Optional Stopping Theorem, we know that $(M_{n\wedge T_U})_{n\geq 0}$ is also a martingale in the same probability space. Then

$$f(x) = E_x[f(X_0)] = E_x[M_{n\wedge T_U}].$$

Since $T_U < \infty$, P_x-a.s. we have that $M_{n\wedge T_U}$ converges P_x-a.s. to M_{T_U} as $n \to \infty$. Also, since f is bounded, M_n is bounded. By the dominated convergence theorem, we have that $E_x[M_{n\wedge T_U}] \to E_x[M_{T_U}]$. Then, it remains to prove that $E_x[M_{T_U}] = E_x[g(X_{T_U})]$, which is equivalent to show that

$$E_x\left[\sum_{k=0}^{T_U-1}(Kf - f)(X_k)\right] = 0. \tag{2}$$

Let us observe that for every $z \in U$

$$\begin{aligned}
Kf(z) &= \int_{\mathbb{Z}^d} f(y)K(z,dy) = \int_{\mathbb{Z}^d} f(y)\frac{1}{2d}\sum_{e\in\mathbb{Z}^d:|e|=1}\delta_{z+e}(dy) \\
&= \frac{1}{2d}\sum_{e\in\mathbb{Z}^d:|e|=1} f(z+e) = f(z)
\end{aligned}$$

where the last equality comes from the fact that f solves $(*)$. Therefore $(Kf - f)$ is constant equal 0 on U. Since $X_k \in U$ for $0 \leq k \leq T_U - 1$, we have that (2) is satisfied.

(c) Let us observe that if $x \notin U$, then $T_U = 0$ and $f(x) = E_x[g(X_0)] = g(x)$. Now, if $x \in U$, we have that there exists $A \subset \Omega$ with $P_x(A) = 1$ such that $T_U(\omega) < \infty$ for all $\omega \in A$. Let us pick $\omega = (x, x_1, x_2, \ldots)$. Since $x \in U$, there exists $k = k(\omega)$ finite such that $1 \leq T_U(\omega) = k$.

Thus $X_{T_U(\omega)} = x_k$. On the other hand, $\theta_1(\omega) = (x_1, x_2, \ldots)$, $T_U(\theta_1(\omega))$ is also equal to k and $X_{T_U(\theta_1(\omega))} = x_k$. This implies that $g(X_{T_U}) = g(X_{T_U}) \circ \theta_1$ P_x-a.s.. Using the Markov property (4.2.55) we get

$$\begin{aligned}
f(x) = E_x[g(X_{T_U})] &= E_x[g(X_{T_U}) \circ \theta_1] \\
\text{(tower property)} &= E_x[E_x[g(X_{T_U}) \circ \theta_1 \mid X_1]] \\
\text{(Markov property)} &= E_x[E^{P_{X_1}}[g(X_{T_U})]] \\
&= E_x[f(X_1)] \\
&= \int_{\mathbb{Z}^d} \delta_x(dx_0) \int_{\mathbb{Z}^d} K(x_0, dx_1) f(x_1) \\
&= \frac{1}{2d} \sum_{e \in \mathbb{Z}^d : |e|=1} f(x+e)
\end{aligned}$$

which is exactly the first condition in $(*)$.