Mathematical Foundations for Finance

Exercise sheet 10

Please hand in your solutions until Wednesday, November 25, 12:00 via the course homepage.

Exercise 10.1 Consider a probability space $(\Omega, \mathcal{F}, P)$ endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$.

(a) Let X be an RCLL $\mathbb{F}$-adapted stochastic process and τ an $\mathbb{F}$-stopping time. Show that if X^τ is an $\mathbb{F}$-martingale, then so is X^σ for any $\mathbb{F}$-stopping time σ with $\sigma \leq \tau$ P-a.s.
Hint: You can use the result that a stopped RCLL martingale is again an RCLL martingale. This result is similar to the result you have proved in Exercise 3.1 (c).

(b) Let M and N be two RCLL local $\mathbb{F}$-martingales. Show that the linear combination $\alpha M + \beta N$ for any $\alpha, \beta \in \mathbb{R}$ is an RCLL local $\mathbb{F}$-martingale as well.
Hint: Make use of the result in (a).

(c) We say that two Brownian motions W^1 and W^2 on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ are *correlated with instantaneous correlation* $\rho \in [-1, 1]$ if for $s \leq t$, the increments $W_t^1 - W_s^1$ and $W_t^2 - W_s^2$ are independent of $\mathcal{F}_s$ and jointly normally distributed with $\mathcal{N}(\mu, \Sigma)$, where

$$\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} t - s & \rho(t-s) \\ \rho(t-s) & t - s \end{pmatrix}.$$

Show that $[W^1, W^2]_t = \rho t$ P-a.s.
Hint: Define $B^\lambda = \lambda(W^1 + W^2)$ with $\lambda \in \mathbb{R}$. Find λ such that B^λ becomes a $(P, \mathbb{F})$-Brownian motion. Then compute $[B^\lambda]$ in terms of W^1 and W^2, using the properties of $[\,\cdot\,,\,\cdot\,]$.

Exercise 10.2 Let M be an RCLL local martingale null at 0 which satisfies $\sup_{0 \leq t \leq T} |M_t| \in L^2$ for some $T \in \mathbb{R}$.

(a) Show that M is a square-integrable martingale on $[0, T]$.
Hint: Dominated convergence theorem.

(b) Let $[M]$ be the square bracket process of M. Show that

$$E\big[[M]_t - [M]_s \,\big|\, \mathcal{F}_s\big] = \operatorname{Var}[M_t - M_s \,|\, \mathcal{F}_s]$$

for all $0 \leq s \leq t \leq T$.
Hint: Recall that $\operatorname{Var}[X \,|\, \mathcal{G}] = E\big[(X - E[X \,|\, \mathcal{G}])^2 \,\big|\, \mathcal{G}\big]$.

Exercise 10.3 On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, consider an adapted stochastic process $X = (X_t)_{t \geq 0}$ null at 0. Assume that X is integrable and has independent stationary increments, i.e. $X_t - X_s$ is independent of $\mathcal{F}_s$ and has the same distribution as X_{t-s} for all $t > s$. (In particular, this is satisfied for any *Lévy process* $L = (L_t)_{t \geq 0}$ with $E[|L_1|] < \infty$).

(a) Which conditions must $(E[X_t])_{t \geq 0}$ satisfy in order to make X a $(P, \mathbb{F})$-supermartingale, a $(P, \mathbb{F})$-submartingale, or a $(P, \mathbb{F})$-martingale?

(b) Assume from now on that X is a square-integrable $(P, \mathbb{F})$-martingale. Prove that we have for all $t, s > 0$ that

$$E\left[X_t^2\right] + E\left[X_s^2\right] = E\left[X_{t+s}^2\right]$$

and deduce that $\left(E\left[X_t^2\right]\right)_{t \geq 0}$ is an increasing process.

(c) Use (b) to prove that $E\left[X_t^2\right] = tE\left[X_1^2\right]$ for all $t \geq 0$.
Hint: Prove the result first for $t = 1/n$ for all $n \in \mathbb{N}$. Deduce that it holds true for all $t \in \mathbb{Q}_+$ and use monotonicity to conclude.

(d) Prove that $\langle X \rangle_t = tE\left[X_1^2\right]$, for all $t \geq 0$.
Hint: Use your result from (c).