# Mathematical Foundations for Finance

## Exercise sheet 11

*Please hand in your solutions until Wednesday, December 2, 12:00 via the course homepage.*

**Exercise 11.1** Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. Assume that $\mathcal{F}_0$ is $P$-trivial and let $W$ be a $(P, \mathbb{F})$-Brownian motion.

(a) Prove that any continuous, adapted process $H$ is predictable and locally bounded.
*Hint 1: Recall that a process $X$ is locally bounded if there exists a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ increasing to infinity such that each $X^{\tau_n}$ is uniformly bounded $P$-a.s.*

(b) Prove that any predictable, locally bounded process $H$ is an element of $L^2_{\mathrm{loc}}(W)$.

(c) Deduce that for any function $f\colon \mathbb{R} \to \mathbb{R}$ in $C^1$, the stochastic integral $\int_0^\cdot f'(W_s)dW_s$ is a continuous local martingale.

(d) Conclude using Itô's formula that $f(W)$ for a given $f \in C^2$ is a continuous local martingale if and only if $\int_0^\cdot f''(W_s)ds = 0$.
*Hint 2: Every local martingale $M$ null at 0 $P$-a.s. which is continuous and of finite variation must be identically* $0$ *$P$-a.s.*

**Exercise 11.2** Let $W = (W_t)_{t\geq 0}$ be a Brownian motion with respect to a probability measure $P$ and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. Using Itô's formula, decide for each of the following processes whether they are local $(P, \mathbb{F})$-martingales or not. Which of them are even true $(P, \mathbb{F})$-martingales?
*Hint: For the martingale property, you only need to look at the processes on $[0, T]$ for some fixed $T > 0$.*

(a) $X_t^{(1)} := \exp\left(\frac{1}{2}\alpha^2 t\right) \cos\left(\alpha(W_t - \beta)\right)$, $t \geq 0$, where $\alpha, \beta \in \mathbb{R}$.

(b) $X_t^{(2)} := \sin W_t - \cos W_t$, $t \geq 0$.

(c) $X_t^{(3)} := W_t^p - ptW_t$, $t \geq 0$, for $p \in \mathbb{N}$ with $p \geq 2$.
*Hint: For any $T > 0$, $\sup_{0\leq t\leq T} W_t$ and $-\inf_{0\leq s\leq T} W_s$ have the same distribution as $|W_T|$.*

**Exercise 11.3** Let $W = (W_t)_{t\geq 0}$ be a Brownian motion with respect to some probability measure $P$ and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. Use Itô's formula to write the following processes as stochastic integrals.

(a) $X_t^{(1)} = W_t^2$.

(b) $X_t^{(2)} = t^2 W_t^3$.

(c) $X_t^{(3)} = \exp(mt + \sigma W_t)$.

(d) $X_t^{(4)} = \cos(t + W_t)$.

(e) $X_t^{(5)} = \log\left(2 + \cos(W_t - t)\right)$.

(f) Let $X$ and $Y$ be two RCLL real-valued $(P, \mathbb{F})$-semimartingales. Define the process $Z = XY$. Write $Z$ as a sum of stochastic integrals.
*Hint: The general Itô's formula for an $\mathbb{R}^d$-valued RCLL semimartingale $X = (X_t)_{t \geq 0}$ and $f \colon \mathbb{R}^d \to \mathbb{R}$ that is in $C^2$ reads:*

$$
\begin{aligned}
f(X_t) =& f(X_0) + \sum_{i=1}^{d} \int_0^t \frac{\partial f}{\partial x^i}(X_{s-}) dX_s^i + \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x^i \partial x^j} f(X_{s-}) d[X^i, X^j]_s \\
&+ \sum_{0<s\leq t} \left( \Delta f(X_s) - \sum_{i=1}^{d} \frac{\partial f}{\partial x^i} f(X_{s-}) \Delta X_s^i - \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x^i \partial x^j} f(X_{s-}) \Delta X_s^i \Delta X_s^j \right).
\end{aligned}
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