Mathematical Foundations for Finance

Exercise 11

Martin Stefanik
ETH Zurich

Local Martingale Properties of Stochastic Integrals

We know from Itô's lemma that a C^2 transformation of any semimartingale is again a semimartingale.

We are often interested in whether these transformations (or stochastic integrals, in which Itô's formula represents these transformations) are square-integrable martingales, martingales or at least local martingales.

- If M is a local martingale and $H \in L^2(M)$, then $(H \cdot M)$ is a martingale in $\mathcal{M}_0^2$; in particular, it is square-integrable.
- If M is a martingale in $\mathcal{M}_0^2$, and H is predictable and bounded, then $(H \cdot M)$ is a martingale in $\mathcal{M}_0^2$; in particular, it is square-integrable.
- If M is a local martingale and $H \in L^2_{\text{loc}}(M)$, then $(H \cdot M)$ is a local martingale in $\mathcal{M}^2_{0,\text{loc}}$; in particular, if $(\tau_n)_{n\in\mathbb{N}}$ is a localizing sequence for $(H \cdot M)$, then $(H \cdot M)^{\tau_n}$ is a square-integrable martingale for all $n \in \mathbb{N}$.
- If M is a local martingale and H is predictable and locally bounded, then $(H \cdot M)$ is local martingale.

Girsanov's Theorem

Theorem 1 (Girsanov's theorem)

Suppose that $Q \overset{loc}{\approx} P$ with a density process Z. If M is a local P-martingale null at 0, then

$$\widetilde{M} := M - \int \frac{1}{Z} d[Z, M]$$

is a local Q-martingale null at 0. As a consequence, every P-semimartingale is also a Q-semimartingale.

- $Q \overset{\text{loc}}{\approx} P$ means that $Q \approx P$ on $\mathcal{F}_T$ for all $T \geq 0$.
- We already know from Itô's lemma that the class of semimartingales is closed under C^2 transformation, i.e. if X is a semimartingale, then $f(X)$ is semimartingale for any $f \in C^2$. Girsanov adds that this property is maintained under a change to any equivalent measure as well.

Girsanov's Theorem

First note that given *any* density process Z, we can write $Z = Z_0\mathcal{E}(L)$, where the local P-martingale is given by

$$L = \int \frac{1}{Z_-} dZ \quad \text{or} \quad dL = \frac{1}{Z_-} dZ.$$

This means that when we want to specify an equivalent measure in terms of a density process, it is satisfactory to consider *only* stochastic exponentials. However, not every stochastic exponential specifies a density process.

Theorem 2 (Girsanov's theorem for continuous density processes)

Suppose that $Q \overset{loc}{\approx} P$ with a continuous density process Z. Write $Z = Z_0\mathcal{E}(L)$. If M is a local P-martingale null at 0, then

$$\widetilde{M} := M - [L, M] = M - \langle L, M\rangle$$

is a local Q-martingale null at zero. Moreover, if W is a P-Brownian motion, then $\widetilde{W}$ is a Q-Brownian motion.

Girsanov's Theorem

- Since the above is just a special case of the former theorem, we must have for any local P-martingale and a *continuous* density process Z that

$$\int \frac{1}{Z} d[Z, M] = [L, M] = \langle L, M \rangle .$$

- We could also write $Z = Z_0 \mathcal{E}(L)$ in the first, more general theorem, but the above simplification would not happen.
- We know that any density process Z can be expressed in terms of a stochastic exponential, but it not the only way to specify an equivalent measure. In particular, we could directly specify the Radon-Nikodým derivative on $\mathcal{F}_T$,

$$\mathcal{D}|_{\mathcal{F}_T} = \frac{dQ|_{\mathcal{F}_T}}{dP|_{\mathcal{F}_T}} .$$

  This can be any $D > 0$ with $E_P[D] = 1$.
- This can be advantageous, since we have seen that for an arbitrary local P-martingale X, $\mathcal{E}(X)$ can be both negative and not necessarily a martingale. $\mathcal{E}(X)$ thus does not specify a density process for any local P-martingale X.

Thank you for your attention!