Mathematical Foundations for Finance

Exercise sheet 12

Please hand in your solutions until Wednesday, December 9, 12:00 via the course homepage.

Exercise 12.1 Let $X = (X_t)_{t\geq 0}$ be a continuous semimartingale null at 0. We define the process

$$Z := \mathcal{E}(X) := e^{X-\frac{1}{2}[X]}.$$

(a) Show via Itô's formula that

$$Z_t = 1 + \int_0^t Z_s dX_s, \quad \forall t \geq 0. \tag{1}$$

Conclude that Z is a continuous local martingale if and only if X is a continuous local martingale.

(b) Show that $Z = \mathcal{E}(X)$ is the only solution to (1) for a given X.
Hint: Let Z' be another solution of (1)*. Compute $\frac{Z'}{Z}$ using Itô's formula.*

(c) Let $Y = (Y_t)_{t\geq 0}$ be another continuous semimartingale null at 0. Prove *Yor's formula*

$$\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}\left(X + Y + [X,Y]\right).$$

Exercise 12.2 Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be a filtered probability space and consider two *independent* Brownian motions $W^1 = (W_t^1)_{t\in[0,T]}$ and $W^2 = (W_t^2)_{t\in[0,T]}$. Let $\widetilde{S}^1 = (\widetilde{S}_t^1)_{t\in[0,T]}$ and $\widetilde{S}^2 = (\widetilde{S}_t^2)_{t\in[0,T]}$ be two *undiscounted* stock price processes with the dynamics

$$\begin{aligned} d\widetilde{S}_t^1 &= \widetilde{S}_t^1\left(\mu_1 dt + \sigma_1 dB_t^1\right), \quad \widetilde{S}_0^1 > 0,\\ d\widetilde{S}_t^2 &= \widetilde{S}_t^2\left(\mu_2 dt + \sigma_2 dB_t^2\right), \quad \widetilde{S}_0^2 > 0, \end{aligned}$$

where $B^1 = W^1$, $B^2 = \alpha W^1 + \sqrt{1-\alpha^2}W^2$ for some $\alpha \in (-1,1)$, $\mu_1, \mu_2 \in \mathbb{R}$ and $\sigma_1, \sigma_2 > 0$.

(a) Find the SDEs satisfied by $X^1 := \frac{\widetilde{S}^2}{\widetilde{S}^1}$ and $X^2 := \frac{\widetilde{S}^1}{\widetilde{S}^2}$, expressed in terms of B^1 and B^2.
Remark: Since $\widetilde{S}^1$ and $\widetilde{S}^2$ have continuous trajectories and satisfy $\widetilde{S}_t^1, \widetilde{S}_t^2 > 0$ for all $t \in [0,T]$ P-a.s., we can choose each of them as *numéraire*.

(b) For $\beta_1, \beta_2 \in \mathbb{R}$, define the continuous local $(P,\mathbb{F})$-martingale $L^{(\beta_1,\beta_2)} := \beta_1 W^1 + \beta_2 W^2$. Show that for all $\beta_1, \beta_2 \in \mathbb{R}$, the stochastic exponential $Z^{(\beta_1,\beta_2)} := \mathcal{E}(L^{(\beta_1,\beta_2)})$ is a true $(P,\mathbb{F})$-martingale on $[0,T]$.

(c) For $\beta_1, \beta_2 \in \mathbb{R}$, define by $dQ^{(\beta_1,\beta_2)} = Z_T^{(\beta_1,\beta_2)} dP$ a probability measure $Q^{(\beta_1,\beta_2)}$ which is equivalent to P on $\mathcal{F}_T$. Fix $\beta_1, \beta_2 \in \mathbb{R}$. Using Girsanov's theorem, show that the two processes $\widetilde{W}_t^1 := W_t^1 - \beta_1 t$ and $\widetilde{W}_t^2 := W_t^2 - \beta_2 t$, $t \in [0,T]$, are local $(Q^{(\beta_1,\beta_2)},\mathbb{F})$-martingales. Conclude that

$$\widetilde{B}^1 := \widetilde{W}^1 \quad \text{and} \quad \widetilde{B}_t^2 := B_t^2 - (\alpha\beta_1 + \sqrt{1-\alpha^2}\beta_2)t, \quad t \in [0,T],$$

are local $(Q^{(\beta_1,\beta_2)},\mathbb{F})$-martingales as well.
Remark: One can show that $\widetilde{W}^1$ and $\widetilde{W}^2$ are *independent* Brownian motions under $Q^{(\beta_1,\beta_2)}$ and correspondingly that $\widetilde{B}^1$ and $\widetilde{B}^2$ are *correlated* Brownian motions under $Q^{(\beta_1,\beta_2)}$.

(d) What conditions on $\beta_1, \beta_2 \in \mathbb{R}$ make the processes X^1 and X^2 $(Q^{(\beta_1,\beta_2)}, \mathbb{F})$-martingales? Can the be martingales simultaneously under the same measure?

Exercise 12.3 Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space with $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. Let $M = (M_t)_{t\geq 0}$ be a local $(P, \mathbb{F})$-martingale and $W = (W_t)_{t\geq 0}$ a $(P, \mathbb{F})$-Brownian motion.

(a) Let $H = (H_t)_{t\geq 0}$ be in $L^2(M)$. Compute $E\left[\int_0^T H_s dM_s\right]$ and $\mathrm{Var}\left[\int_0^T H_s dM_s\right]$. How do the expressions look for $M := W$?

(b) Let $H_s := \exp(-4s)$. Show that $\int_0^T H_s dW_s$ is in fact normally distributed. What are the mean and the variance of this normal distribution? How would the result change if $H : \mathbb{R} \to \mathbb{R}$ were an arbitrary (deterministic) continuous function?
Hint 1: Use the dominated convergence theorem for stochastic integrals from the lecture notes.
Hint 2: If $X_n \sim \mathcal{N}(\mu_n, \sigma_n^2)$, $X_n \to X$ in probability, $\mu_n \to \mu$ and $\sigma_n^2 \to \sigma^2 > 0$, then $X \sim \mathcal{N}(\mu, \sigma^2)$.

(c) By coming up with a counterexample, show that the normality of $\int_0^T H_s dW_s$ from (b) does not hold for an arbitrary continuous process $H \in L^2(W)$.