Mathematical Foundations for Finance

Exercise sheet 13

Please hand in your solutions until Wednesday, December 16, 12:00 via the course homepage.

Exercise 13.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. Let $N = (N_t)_{t\geq 0}$ be a $(P, \mathbb{F})$-Poisson process with parameter $\lambda > 0$. Let $\widetilde{\lambda} > 0$ and define a process $S = (S_t)_{t\geq 0}$ by

$$S_t := e^{(\lambda-\widetilde{\lambda})t}\left(\frac{\widetilde{\lambda}}{\lambda}\right)^{N_t}.$$

(a) Let $X = (X_t)_{t\geq 0}$ be given by $X_t = \alpha t + \beta N_t$ for some $\alpha, \beta \in \mathbb{R}$. Show that we have for any C^2-function $f\colon \mathbb{R} \to \mathbb{R}$ that

$$f(X_t) = f(0) + \alpha \int_0^t f'(X_{s-})ds + \sum_{0<s\leq t}\Big(f(X_s) - f(X_{s-})\Big).$$

Hint: Use the results we have shown for Poisson processes in Exercise 9.2.

(b) Show that P-a.s. for all $t > 0$, we have that

$$\Delta S_t = \frac{\widetilde{\lambda} - \lambda}{\lambda} S_{t-}\Delta N_t.$$

(c) Show that P-a.s. for all $t \geq 0$, we have that

$$S_t = 1 + \int_0^t \frac{\widetilde{\lambda} - \lambda}{\lambda} S_{u-} d\widetilde{N}_u,$$

where $\widetilde{N} = (\widetilde{N})_{t\geq 0}$ is a compensated Poisson process with parameter λ.
Hint: Use your result from (a).

(d) Deduce that S is a local $(P, \mathbb{F})$-martingale. Show that it is even a true $(P, \mathbb{F})$-martingale.
Hint: Show that $\sup_{0\leq t\leq T} |S_t|$ *is integrable for each* $T > 0$.

Exercise 13.2 Let $T > 0$ denote a fixed time horizon and $W = (W_t)_{t\in[0,T]}$ a Brownian motion on some probability space $(\Omega, \mathcal{F}, P)$. Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by W and augmented by the P-nullsets in $\sigma(W_s, s \leq T)$. Consider the Black–Scholes model, where the undiscounted bank account price process $\widetilde{S}^0 = (\widetilde{S}^0_t)_{t\in[0,T]}$ and the undiscounted stock price process $\widetilde{S}^1 = (\widetilde{S}^1_t)_{t\in[0,T]}$ are given by

$$d\widetilde{S}^0_t = \widetilde{S}^0_t r dt \quad \text{and} \quad d\widetilde{S}^1_t = \widetilde{S}^1_t\left(\mu dt + \sigma dW_t\right), \tag{1}$$

where $r, \mu \in \mathbb{R}$ and $\sigma > 0$ as well as $\widetilde{S}^0_0 = 1$ and $\widetilde{S}^1_0 > 0$ are deterministic.

(a) Prove using Itô's formula that the discounted stock price process $S^1 = \widetilde{S}^1/\widetilde{S}^0$ solves

$$dS^1_t = S^1_t\big((\mu - r)dt + \sigma dW_t\big). \tag{2}$$

(b) Prove using Itô's formula that

$$S_t^1 = S_0^1 \exp\left(\sigma W_t + \left(\mu - r - \frac{1}{2}\sigma^2\right) t\right), \quad \text{for } t \in [0,T],$$

i.e., show that the process $\left(S_0^1 \exp\left(\sigma W_t + \left(\mu - r - \frac{1}{2}\sigma^2\right) t\right)\right)_{t\in[0,T]}$ solves (2).

(c) Let $L^\lambda := -\lambda W$ and $Z^\lambda := \mathcal{E}(L^\lambda)$. Prove that the process $W^\lambda := (W_t + \lambda t)_{t\in[0,T]}$ is a Brownian motion on $[0,T]$ under the measure Q_λ given by $\frac{dQ_\lambda}{dP} := Z_T^\lambda$.

(d) Prove that for the right choice of λ, the discounted stock price process S^1 is a Q_λ-martingale. *Hint: Rewrite* $\sigma W_t + \left(\mu - r - \frac{1}{2}\sigma^2\right) t$ *as a function of* $W_t^\lambda, t, \sigma, \mu,$ *and* r.

Exercise 13.3 Let $T > 0$ denote a fixed time horizon and let $W = (W_t)_{t\in[0,T]}$ be a Brownian motion on some probability space $(\Omega, \mathcal{F}, P)$. Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by W and augmented by the P-nullsets in $\sigma(W_s, 0 \le s \le T)$. Consider the Black–Scholes model, where the undiscounted bank account price process $\widetilde{S}^0 = (\widetilde{S}_t^0)_{t\in[0,T]}$ and the undiscounted stock price process $\widetilde{S}^1 = (\widetilde{S}_t^1)_{t\in[0,T]}$ are given by

$$\frac{d\widetilde{S}_t^0}{\widetilde{S}_t^0} = r\,dt \quad \text{and} \quad \frac{d\widetilde{S}_t^1}{\widetilde{S}_t^1} = \mu dt + \sigma dW_t,$$

with $r, \mu \in \mathbb{R}$ and $\sigma > 0$ as well as $\widetilde{S}_0^0 = 1$ and $\widetilde{S}_0^1 > 0$ deterministic. Using the notation of the previous exercise, denote $Q^* := Q_{\lambda^*}$, where λ^* is the unique value of λ making Q_λ an equivalent martingale measure for $S^1 := \widetilde{S}^1/\widetilde{S}^0$.
Hint: If you did not find λ^* *in Exercise 13.2 (d), you can use that* $\lambda^* = \frac{\mu - r}{\sigma}$.

(a) Hedge the discounted *square option*, i.e., find a self-financing strategy $\varphi \mathrel{\widehat{=}} (V_0, \vartheta)$ such that

$$V_0 + \int_0^T \vartheta_u dS_u^1 = \frac{\left(\widetilde{S}_T^1\right)^2}{\widetilde{S}_T^0}.$$

Hint: Look for a representation result under Q^**, not under* P.

(b) Hedge the discounted *inverted option*, i.e., find a self-financing strategy $\varphi \mathrel{\widehat{=}} (\overline{V}_0, \overline{\vartheta})$ such that

$$\overline{V}_0 + \int_0^T \overline{\vartheta}_u dS_u^1 = \frac{1}{\widetilde{S}_T^0 \widetilde{S}_T^1}.$$