Mathematical Foundations for Finance

Exercise sheet 14

This exercise sheet will not be corrected. Please, do not hand in.

Exercise 14.1 We use the techniques developed in the course in a slightly different setting. Consider a financial market $(\widetilde{S}^0, \widetilde{S}^1)$ on a probability space $(\Omega, \mathcal{F}, P)$ endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. Let W^1 and W^2 be two $(P, \mathbb{F})$-Brownian motions with a constant instantaneous correlation $\rho \in (-1, 1)$, and let the dynamics of $\widetilde{S}^0$ and $\widetilde{S}^1$ be described by the SDEs

$$\begin{aligned} d\widetilde{S}^0_t &= \widetilde{S}^0_t r_t dt, \\ dr_t &= \theta(\alpha - r_t)dt + \eta dW^1_t, \\ d\widetilde{S}^1_t &= \widetilde{S}^1_t \left(r_t dt + \sigma dW^2_t\right), \end{aligned}$$

where $\sigma, \eta > 0$ and $\theta, \alpha \in \mathbb{R}$ as well as $\widetilde{S}^0_0 = 1$, $\widetilde{S}^1_0 > 0$ and $r_0 \in \mathbb{R}$ are all constant. This is the Black–Scholes model with a stochastic interest rate.

(a) By applying Itô's formula to some function $f \in C^2$ and the continuous semimartingale $\widetilde{S}^0$, show that the solution to the first SDE is given by

$$\widetilde{S}^0_t = \exp\left(\int_0^t r_s ds\right).$$

(b) By applying Itô's formula to the function $f(x, t) = xe^{\theta t}$ and the continuous semimartingale $(r_t, t)_{t\geq 0}$, show that the solution to the second SDE is given by

$$r_t = r_0 e^{-\theta t} + \alpha(1 - e^{-\theta t}) + \eta e^{-\theta t} \int_0^t e^{\theta s} dW_s.$$

The solution to this SDE is called *Ornstein–Uhlenbeck process* and is an important process when it comes to interest rate modelling.

(c) Show that the discounted price processes $S^0 := \widetilde{S}^0/\widetilde{S}^0$ and $S^1 := \widetilde{S}^1/\widetilde{S}^0$ are $(P, \mathbb{F})$-martingales, i.e. the market $(\widetilde{S}^0, \widetilde{S}^1)$ is arbitrage-free and we can use P as our pricing measure.

Exercise 14.2 Let $W = (W_t)_{t\geq 0}$ be a Brownian motion with respect to a probability measure P and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. We know from the lecture that if $H \in L^2_{\text{loc}}(W)$, then $\int H dW$ is a local $(P, \mathbb{F})$-martingale. The purpose of this exercise is to study when $\int H dW$ is a even a true $(P, \mathbb{F})$-martingale. Show that $\int H dW$ is a $(P, \mathbb{F})$-martingale on $[0, \infty)$ if

(a) $\int H dW$ is a $(P, \mathbb{F})$-martingale on $[0, T]$ for every $0 \leq T < \infty$.

(b) $\int H dW$ has a majorant in $L^1(P)$ on $[0, T]$, i.e.

$$\left|\int_0^t H_s dW_s\right| \leq X \quad \text{for all } t \in [0, T],$$

where $X \in L^1(P)$.

(c) $H \in L^2(W^T)$ for any $T \geq 0$, i.e. if $E\left[\int_0^T H_s^2 ds\right] < \infty$ for any $T \geq 0$.

Exercise 14.3 Let $(\Omega, \mathcal{F}, P)$ be a probability space with a Brownian motion $W = (W_t)_{t \in [0,T]}$. Let $\mathbb{F} := \mathbb{F}^W$ be the (augmented) filtration generated by W. Consider the discounted price process $S = (S_t)_{t \in [0,T]}$ with dynamics

$$dS_t = \sigma(t, S_t) dW_t, \quad S_0 > 0,$$

where $\sigma \colon [0,T] \times \mathbb{R} \to (0, \infty)$ is a continuous and bounded function. One can show that S is well defined and P is the unique EMM for S. Let $h \colon \mathbb{R} \to [0, \infty)$ be a fixed continuous and bounded function. We consider the partial differential equation (PDE)

$$\begin{cases} \frac{\partial}{\partial t} v(t,x) + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2}{\partial x^2} v(t,x) = 0, & x \in (0,T) \times \mathbb{R}, \\ v(T,x) = h(x), & x \in \mathbb{R}. \end{cases} \tag{1}$$

Suppose that there exists a $C^{1,2}$ solution $v \colon [0,T] \times \mathbb{R} \to \mathbb{R}$ of (1) with the additional property that

$$\left| \frac{\partial}{\partial x} v(t,x) \right| \leq C(1 + |x|), \quad (t,x) \in [0,T] \times \mathbb{R},$$

for some constant $C > 0$. Show that $V_t^* := v(t, S_t)$, $t \in [0,T]$, is the price at time t of the discounted European contingent claim $h(S_T)$.