Mathematical Foundations for Finance

Exercise sheet 2

Please hand in your solutions until Wednesday, September 30, 12:00 via the course homepage.

Exercise 2.1 Let us assume the basic multiplicative model for our financial market $(\widetilde{S}^0, \widetilde{S}^1)$. We start on a probability space $(\Omega, \mathcal{F}, P)$ with random variables $r_1, \ldots, r_T > -1$ and $Y_1, \ldots, Y_T > 0$ for a $T \in \mathbb{N}$. Define for $k = 0, \ldots, T$

$$\widetilde{S}^0_k := \prod_{j=1}^{k} (1 + r_j), \quad \widetilde{S}^1_k := S^1_0 \prod_{j=1}^{k} Y_j,$$

with a constant $S^1_0 > 0$.

(a) A natural filtration to use in this model is the filtration $\mathbb{F}' = (\mathcal{F}'_k)_{k=0,\ldots,T}$ generated by $Y = (Y_k)_{k=1,\ldots,T}$ and $r = (r_k)_{k=1,\ldots,T}$, i.e. the one given by

$$\begin{aligned} \mathcal{F}'_0 &= \{\emptyset, \Omega\}, \\ \mathcal{F}'_k &= \sigma(Y_1, \ldots, Y_k, r_1, \ldots, r_k) \quad \text{for } k = 1, \ldots, T. \end{aligned}$$

Show that if one assumes r to be predictable with respect $\mathbb{F}'$, then we have that $\mathcal{F}'_k = \mathcal{F}_k := \sigma(\widetilde{S}^1_0, \widetilde{S}^1_1, \ldots, \widetilde{S}^1_k)$ for all $k = 0, \ldots, T$.
Hint: if $\mathcal{A}$ and $\mathcal{B}$ are two collections of subsets of Ω, then $\sigma(\mathcal{A} \cup \mathcal{B}) = \sigma(\sigma(\mathcal{A}) \cup \sigma(\mathcal{B}))$. To get some extra practice, try to prove the hint as well.

(b) Recall that we call a strategy $\varphi = (\varphi^0, \vartheta)$ self-financing if its discounted cost process $C(\varphi)$ is constant over time. Show that the notion of self-financing strategy does not depend on discounting: If $D = (D_k)_{k=0,\ldots,T}$ is any strictly positive adapted process (for instance $D = \widetilde{S}^0$) and $\bar{S}^i_k := S^i_k D_k$ for all $k = 0, \ldots, T$ and $i = 0, 1$ then the discounted cost process $C(\varphi)$ is constant over time if and only if the undiscounted cost process $\bar{C}(\varphi)$, determined by

$$\Delta \bar{C}_{k+1}(\varphi) := \bar{C}_{k+1}(\varphi) - \bar{C}_k(\varphi) = (\varphi^0_{k+1} - \varphi^0_k)\bar{S}^0_k + (\vartheta_{k+1} - \vartheta_k)\bar{S}^1_k,$$

is constant over time.

(c) Use the result in (b) to conclude that the notion of self-financing strategy is numeraire-invariant, i.e. that it does not matter for this definition whether the discounted price processes are defined as $S^0 := \widetilde{S}^0/\widetilde{S}^0$ and $S^1 := \widetilde{S}^1/\widetilde{S}^0$, or $\bar{S}^0 := \widetilde{S}^0/\widetilde{S}^1$ and $\bar{S}^1 := \widetilde{S}^1/\widetilde{S}^1$.

Exercise 2.2 Consider for a finite time horizon $T \geq 2$ a financial market $(\widetilde{S}^0, \widetilde{S}^1)$ consisting of a bank account and one stock defined on a probability space $(\Omega, \mathcal{F}, P)$. Assume that $\widetilde{S}^1_0 = 1$ and $\widetilde{S}^1_k > 0$ P-a.s. for all $k = 0, \ldots, T$. Fix thresholds $0 < \ell < 1 < u$ and define

$$\begin{aligned} \rho(\omega) &:= \inf\{k = 0, \ldots, T : S^1_k(\omega) \leq \ell\} \wedge T, \\ \tau(\omega) &:= \inf\{k = \rho(\omega), \ldots, T : S^1_k(\omega) \geq u\} \wedge T, \end{aligned}$$

where $\inf \emptyset = +\infty$ as usual. Moreover, for $k = 0, \ldots, T$, define

$$\vartheta_k(\omega) := \mathbb{1}_{\{\rho(\omega) < k \leq \tau(\omega)\}}.$$

Finally define the filtration $\mathbb{F} = (\mathcal{F}_k)_{0 \leq k \leq T}$ by $\mathcal{F}_0 = \{\emptyset, \Omega\}$, and $\mathcal{F}_k = \sigma(\widetilde{S}^1_i,\ i \leq k)$.

(a) Show that ρ and τ are *stopping times*, i.e. that for all $k = 0, \dots, T$, we have

$$\{\rho \leq k\}, \{\tau \leq k\} \in \mathcal{F}_k\,.$$

(b) Show that ϑ is a predictable process with $\vartheta_0 = \vartheta_1 = 0$.

(c) Construct φ^0 such that $\varphi = (\varphi^0, \vartheta)$ is a self-financing strategy with $V_0(\varphi) = 0$ and derive a formula for the (discounted) value process $V(\varphi)$ involving only the discounted stock price S^1 and the stopping times ρ and τ.

(d) Describe the trading strategy φ in words.

Exercise 2.3 The following is an exercise on the so-called (functional) *monotone class theorem*. This theorem is very useful for proving results for complicated classes of measurable functions (random variables) using simpler classes of measurable functions for which the desired result is easier to prove. We recall the theorem below.

Theorem (Monotone class theorem)**.** *Let $B(\Omega)$ be the set of bounded functions $f\colon \Omega \to \mathbb{R}$. Let $\mathcal{M} \subseteq B(\Omega)$ such that $fg \in \mathcal{M}$ for $f, g \in \mathcal{M}$ (i.e. $\mathcal{M}$ is closed under multiplication). Lastly, let $\mathcal{H}$ be a real vector subspace of $B(\Omega)$ such that*

- *$\mathcal{M} \subseteq \mathcal{H}$.*
- *$\mathcal{H}$ contains the constant function 1.*
- *For every non-decreasing (uniformly) bounded sequence $(f_n)_{n\in\mathbb{N}}$ of non-negative functions in $\mathcal{H}$, we have that $f := \lim_{n\to\infty} f_n$ is in $\mathcal{H}$ (i.e. $\mathcal{H}$ is closed under bounded monotone limits).*

Then $\mathcal{H}$ contains all bounded $\sigma(\mathcal{M})$-measurable functions.

Now let $(E, \mathcal{E}, \mu)$ and $(F, \mathcal{F}, \nu)$ be finite measure spaces (that is, $\mu(E), \nu(F) < \infty$).

(a) Using the monotone class theorem, show that for every bounded $\mathcal{E} \otimes \mathcal{F}$-measurable function $f\colon E \times F \to \mathbb{R}$, the function $E \ni x \mapsto \int_F f(x, y) d\nu(y)$ is $\mathcal{E}$-measurable.
Hint: Recall that the product σ-algebra $\mathcal{E} \otimes \mathcal{F}$ is defined as the σ-algebra generated by the sets of the form $A \times B$, $A \in \mathcal{E}$ and $B \in \mathcal{F}$. Use the sets of this form to define a suitable class $\mathcal{M}$.

(b) Show that for every $\mathcal{E} \otimes \mathcal{F}$-measurable, $\mu \otimes \nu$-integrable function $f\colon E \times F \to \mathbb{R}$, the function $E \ni x \mapsto \int_F f(x, y) d\nu(y)$ is $\mathcal{E}$-measurable.
Hint: Note that every non-negative measurable function can be written as monotone limit of a sequence of simple functions.

(c) Using the monotone class theorem, show that for every non-negative $\mathcal{E} \otimes \mathcal{F}$-measurable function $f\colon E \times F \to \mathbb{R}$ we have that (Fubini–Tonelli theorem for finite measure spaces)

$$\int_E \int_F f(x, y) d\nu(y) d\mu(x) = \int_F \int_E f(x, y) d\mu(x) d\nu(y). \tag{1}$$

Hint: Start by proving the result for bounded $\mathcal{E} \otimes \mathcal{F}$-measurable functions similarly to (a).