Mathematical Foundations for Finance

Exercise sheet 3

Please hand in your solutions until Wednesday, October 7, 12:00 via the course homepage.

Exercise 3.1 Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space with $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,T}$ and let $X = (X_k)_{k=0,1,\dots,T}$ be a martingale with respect to $\mathbb{F}$ and P.

(a) Show that for any bounded, convex function $f \colon \mathbb{R} \to \mathbb{R}$, the process $f(X) = (f(X_k))_{k=0,1,\dots,T}$ is a submartingale with respect to $\mathbb{F}$ and P. What can you say if f is not bounded?
Hint: Any convex function on $\mathbb{R}^n$ is continuous on the interior of the set where it is finite.

(b) Let $\vartheta = (\vartheta_k)_{k=0,1,\dots,T}$ with $\vartheta_0 = 0$ be a bounded, nonnegative, $\mathbb{F}$-predictable process. Show that the stochastic integral process $\vartheta \bullet f(X) := \vartheta \bullet \big(f(X)\big)$ defined by

$$\vartheta \bullet f(X)_k = \sum_{j=1}^{k} \vartheta_j \Delta f(X_j) = \sum_{j=1}^{k} \vartheta_j \big(f(X_j) - f(X_{j-1})\big)$$

is a submartingale with respect to $\mathbb{F}$ and P.
Hint: We have proved in the lecture that the stochastic integral process of a similar predictable process with respect to a martingale is martingale.

(c) Let τ be a stopping time with respect to $\mathbb{F}$ and define the stopped process $f(X)^\tau = (f(X)^\tau_k)_{k=0,1,\dots,T}$ by $f(X)^\tau_k = f(X_{k\wedge\tau})$. Show that $f(X)^\tau$ is a submartingale with respect to $\mathbb{F}$.
Hint: Try to express the stopped process as an appropriate stochastic integral process.

(d) Let $Y = (Y_k)_{k=0,1,\dots,T}$ be an adapted, integrable process. Show that Y is a martingale if and only if $E[Y_{k+1} \,|\, \mathcal{F}_k] = Y_k$ P-a.s. for $k = 0, 1, \dots, T-1$.

Exercise 3.2 Let $(\widetilde{S}^0, \widetilde{S}^1)$ be a market modelled by a *binomial model*. More precisely, let the undiscounted price processes of the assets in our market be defined by

$$\widetilde{S}^0_k = (1+r)^k \quad \text{for } k = 0, 1, \dots, T,$$
$$\frac{\widetilde{S}^1_{k+1}}{\widetilde{S}^1_k} = Y_{k+1} \quad \text{for } k = 0, 1, \dots, T-1,$$

where the Y_k are i.i.d. random variables taking values $1+u$ with probability $p \in (0,1)$ and $1+d$ with probability $1-p$. Assume furthermore that $u > d > -1$ and $r > -1$.

(a) Suppose that $r \leq d$. Show that in this case, the market $(\widetilde{S}^0, \widetilde{S}^1)$ admits *arbitrage* by explicitly constructing an *arbitrage opportunity*.

(b) Suppose that $r \geq u$. Show that also in this case, the market $(\widetilde{S}^0, \widetilde{S}^1)$ admits *arbitrage* by explicitly constructing an *arbitrage opportunity*.

Exercise 3.3 Let $(\widetilde{S}^0, \widetilde{S}^1)$ be a *trinomial model*. This is, like the binomial model, a special case of a *multinomial model*, and the distribution of Y_k under P is given by

$$Y_k = \begin{cases} 1+d & \text{with probability } p_1, \\ 1+m & \text{with probability } p_2, \\ 1+u & \text{with probability } p_3 \end{cases}$$

where p_1, p_2, $p_3 > 0$, $p_1 + p_2 + p_3 = 1$ and $-1 < d < m < u$. Here d, m and u are mnemonics for *down*, *middle* and *up*. Recall that the filtration $\mathbb{F}$ we consider is generate by Y.

(a) Assume that $d = -0.5$, $m = 0$, $u = 0.25$ and $r = 0$. For $T = 1$, consider an arbitrary self-financing strategy $\varphi \mathrel{\widehat{=}} (V_0, \theta)$. Show that if the total gain $G_1(\theta)$ at time $T = 1$ is nonnegative P-a.s., then

$$P[G_1(\theta) = 0] = 1.$$

What does this property imply?

(b) Show that S^1 is arbitrage-free by constructing an *equivalent martingale measure* (EMM) for S^1.
Hint: A probability measure Q equivalent to P on $\mathcal{F}_1$ can be uniquely described by a probability vector $(q_1, q_2, q_3) \in (0,1)^3$, where $q_k = Q[Y_1 = 1 + y_k]$, $k = 1, 2, 3$, using the notation $y_1 := d$, $y_2 := m$ and $y_3 := u$. A probability vector in $\mathbb{R}^n$, $n \in \mathbb{N}$, is a nonnegative vector in $\mathbb{R}^n$ whose coordinates sum up to 1.

(c) Assume now that $d = -0.01$, $m = 0.01$, $u = 0.03$ and $r = 0.01$. For $T = 2$, give a parametrisation of all EMMs for S^1.
Hint: A probability measure Q equivalent to P on $\mathcal{F}_2$ can be uniquely described by four probability vectors (q_1, q_2, q_3), $(q_{j,1}, q_{j,2}, q_{j,3}) \in (0,1)^3$, $j = 1, 2, 3$, where $q_j = Q[Y_1 = 1 + y_j]$ and $q_{j,k} = Q[Y_2 = 1 + y_k \,|\, Y_1 = 1 + y_j]$, $j, k = 1, 2, 3$, using the notation $y_1 := d$, $y_2 := m$ and $y_3 := u$.