Mathematical Foundations for Finance

Exercise sheet 6

Please hand in your solutions until Wednesday, October 28, 12:00 via the course homepage.

Exercise 6.1 Consider a financial market in finite discrete time on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with undiscounted prices $\widetilde{S}^0$ and $\widetilde{S}$ and discounted prices 1 and $S = \widetilde{S}/\widetilde{S}^0$. An arbitrage opportunity in the undiscounted market is a self-financing strategy φ with $\widetilde{V}(\varphi) \geq -a$ P-a.s. for some $a \geq 0$ (admissibility), $\widetilde{V}_0(\varphi) = 0$, $\widetilde{V}_T(\varphi) \geq 0$ P-a.s. and $P\big[\widetilde{V}_T(\varphi) > 0\big] > 0$.

(a) Show that $(\widetilde{S}^0, \widetilde{S})$ is free of arbitrage if and only if $(1, S)$ is.
Hint: Remember that in finite discrete time, we have $NA \iff NA'$.

(b) Construct an example where $\widetilde{S}$ admits an EMM, but is not arbitrage-free. Does S then admit an EMM? What can you say about $\widetilde{S}^0$ for any such example?

(c) In your example, construct explicitly an arbitrage opportunity for the undiscounted market.

(d) Try to provide some intuition for why the existence of an EMM for $\widetilde{S}$ does not imply NA, when we know that the existence of an EMM for S does.

Exercise 6.2 Let $(\Omega, \mathcal{F}, \mathbb{F}, P, \widetilde{S}^0, \widetilde{S}^1)$ be our canonical setup for a one-period trinomial model in which the evolution of $(\widetilde{S}^0, \widetilde{S}^1)$ is given by

$$\widetilde{S}^1_0 = S^1_0 = 80, \quad \widetilde{S}^1_1 = \begin{cases} 120 & \text{with probability } p_1 = 0.2, \\ 90 & \text{with probability } p_2 = 0.3, \\ 60 & \text{with probability } p_3 = 0.5 \end{cases}$$
$$\widetilde{S}^0_0 = 1, \quad \widetilde{S}^0_1 = 1 + 0.05.$$

(a) Check if the market is arbitrage-free by finding at least one EMM for $S^1 = \widetilde{S}^1/\widetilde{S}^0$.

(b) Find the set of all EMMs for S^1.

(c) Compute $E_Q\big[\widetilde{C}/\widetilde{S}^0_1\big]$ for all $Q \in P_e(S^1)$, where $\widetilde{C}$ is the (undiscounted) payoff of a European call option with maturity $T = 1$ and strike price $\widetilde{K} = 80$, i.e. $\widetilde{C}(\omega) = \big(\widetilde{S}^1_1(\omega) - 80\big)^+$.

(d) An undiscounted payoff $H \in L^0_+(\mathcal{F}_T)$ is attainable if it can be written as $\widetilde{H} = \widetilde{V}(\varphi)$ for some self-financing strategy φ which is admissible in the undiscounted market. Determine whether $\widetilde{C}$ as given in (c) is attainable.

(e) Find the set of all attainable payoffs $\widetilde{H} \in L^0_+(\mathcal{F}_T)$.
Hint: Every payoff is characterized by the values it takes on the atoms of $\mathcal{F}_T$*. The set of all attainable payoffs can be identified with the set of solutions to a linear system.*

Exercise 6.3 Consider a measurable space $(\Omega, \mathcal{F})$ with random variables $Y_1, \dots, Y_T > 0$, $T \in \mathbb{N}$. The random variables define a multiplicative model for $\widetilde{S}^1 = \big(\widetilde{S}^1_k\big)_{k=0,1,\dots,T}$ by taking

$$\widetilde{S}^1_k := s_0 \prod_{i=1}^{k} Y_i, \quad \text{for some } s_0 > 0.$$

Endow $(\Omega, \mathcal{F})$ with the filtration $\mathbb{F}$ generated by $\widetilde{S}^1$. Furthermore, let $\widetilde{S}^0 \equiv 1$, which corresponds to zero interest rates and gives $S^1 \equiv \widetilde{S}^1$. Now consider the self-financing "buy low, sell high" strategy $\varphi \mathrel{\widehat{=}} (0, \vartheta)$ from Exercise 2.2 with $\vartheta = (\vartheta_k)_{k=0,1,\dots,T}$ given by $\vartheta_k = \mathbb{1}_{\{\rho < k \leq \tau\}}$, where ρ and τ are the $\mathbb{F}$-stopping times defined by

$$\begin{aligned}\rho(\omega) &:= \inf\{k = 0, \dots, T : S^1_k(\omega) \leq \ell\} \wedge T, \\ \tau(\omega) &:= \inf\{k = \rho(\omega), \dots, T : S^1_k(\omega) \geq u\} \wedge T,\end{aligned}$$

for $0 < \ell < s_0 < u$. Recall that the strategy's discounted value process $V(\varphi) = \big(V_k(\varphi)\big)_{k=0,1,\dots,T}$ is given by

$$V_k(\varphi) = S^1_{\tau \wedge k} - S^1_{\rho \wedge k}.$$

Note that nothing defined up to this point requires to have a probability measure on $(\Omega, \mathcal{F})$.

(a) Give conditions on a probability measure P on $(\Omega, \mathcal{F})$, expressed via the process S^1, that are equivalent to φ *not* being an *arbitrage opportunity*.
Hint: Partition Ω into sets on which it is easier to see how φ behaves.

(b) Express the conditions on P from (a) as conditions involving the random variables $Y_1, \dots, Y_T$.