Mathematical Foundations for Finance

Exercise sheet 7

Please hand in your solutions until Wednesday, November 4, 12:00 via the course homepage.

Exercise 7.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space endowed with a filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,T}$, and let τ be an $\mathbb{F}$-stopping time. We define

$$\mathcal{F}_\tau := \{A \in \mathcal{F} \colon A \cap \{\tau \leq k\} \in \mathcal{F}_k \text{ for all } k = 0, 1, \dots, T\}.$$

(a) Show that $\mathcal{F}_\tau$ is a σ-algebra.

(b) Suppose σ, τ are two $\mathbb{F}$-stopping times with $\sigma(\omega) \leq \tau(\omega)$ for all $\omega \in \Omega$. Show that $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$. Conclude that if $\tau \equiv k_0$ for a fixed $k_0 \in \{0, 1, \dots, T\}$, then we have $\mathcal{F}_\tau = \mathcal{F}_{k_0}$.

(c) If τ, σ are two $\mathbb{F}$-stopping times, show that $\tau \vee \sigma$ and $\tau \wedge \sigma$ are $\mathbb{F}$-stopping times, and $\mathcal{F}_\tau \cap \mathcal{F}_\sigma = \mathcal{F}_{\tau \wedge \sigma}$. Moreover, show that $\{\sigma \leq \tau\} \in \mathcal{F}_{\tau \wedge \sigma}$ and $\{\sigma = \tau\} \in \mathcal{F}_{\tau \wedge \sigma}$.

(d) Show that for a random variable $Y \in L^0_+(\mathcal{F})$, we have

$$E\left[Y \,|\, \mathcal{F}_\tau\right] \mathbb{1}_{\{\tau = k\}} = E\left[Y \,|\, \mathcal{F}_k\right] \mathbb{1}_{\{\tau = k\}} \; P\text{-a.s. for all } k \in \{0, 1, \dots, T\},$$

i.e. that $E\left[Y \,|\, \mathcal{F}_\tau\right] = E\left[Y \,|\, \mathcal{F}_k\right]$ P-a.s. on the set $\{\tau = k\}$ or, equivalently,

$$E\left[Y \,|\, \mathcal{F}_\tau\right] = \sum_{k=0}^{T} \mathbb{1}_{\{\tau = k\}} E\left[Y \,|\, \mathcal{F}_k\right] \;\; P\text{-a.s.}$$

Exercise 7.2 Let $(\widetilde{S}^0, \widetilde{S}^1)$ be an *arbitrage-free* financial market with time horizon T and assume that the bank account process $\widetilde{S}^0 = (\widetilde{S}^0_k)_{k=0,1,\dots,T}$ is given by $\widetilde{S}^0_k = (1+r)^k$ for a constant $r \geq 0$. Denote the set of all EMMs for $S^1 = \widetilde{S}^1 / \widetilde{S}^0$ by $\mathbb{P}_e(S^1)$. Fix a $\widetilde{K} > 0$. The undiscounted payoff of a *European call option* on $\widetilde{S}^1$ with strike $\widetilde{K}$ and maturity $k \in \{1, \dots, T\}$ is denoted by $\widetilde{C}^E_k$ and given by

$$\widetilde{C}^E_k = \big(\widetilde{S}^1_k - \widetilde{K}\big)^+,$$

whereas the undiscounted payoff of an *Asian call option* on $\widetilde{S}^1$ with strike $\widetilde{K}$ and maturity $k \in \{1, \dots, T\}$ is denoted by $\widetilde{C}^A_k$ and given by

$$\widetilde{C}^A_k := \left(\frac{1}{k} \sum_{j=1}^{k} \widetilde{S}^1_j - \widetilde{K} \right)^+.$$

(a) Fix a $Q \in \mathbb{P}_e(S^1)$ and show that the function $\{1, \dots, T\} \to \mathbb{R}_+$, $k \mapsto E_Q\left[\frac{\widetilde{C}^E_k}{\widetilde{S}^0_k}\right]$ is increasing.
Hint: Use Jensen's inequality for conditional expectations.

(b) Fix a $Q \in \mathbb{P}_e(S^1)$ and show that for all $k = 1, \dots, T$, we have

$$E_Q\left[\frac{\widetilde{C}^A_k}{\widetilde{S}^0_k}\right] \leq \frac{1}{k} \sum_{j=1}^{k} E_Q\left[\frac{\widetilde{C}^E_j}{\widetilde{S}^0_j}\right].$$

(c) Fix a $Q \in \mathbb{P}_e(S^1)$ and deduce that for all $k = 1, \ldots, T$, we have

$$E_Q\left[\frac{\widetilde{C}^A_k}{\widetilde{S}^0_k}\right] \le E_Q\left[\frac{\widetilde{C}^E_k}{\widetilde{S}^0_k}\right].$$

Interpret this inequality.

Exercise 7.3 Let $(\widetilde{S}^0, \widetilde{S}^1)$ follow a binomial model with $\widetilde{S}^1_0 = 1$, $u > r > d > -1$ and $T \in \mathbb{N}$. Denote by $(\widehat{S}^0, \widehat{S}^1)$ the market discounted with $\widetilde{S}^1$, i.e.

$$\widehat{S}^0 := \frac{\widetilde{S}^0}{\widetilde{S}^1} \quad \text{and} \quad \widehat{S}^1 := \frac{\widetilde{S}^1}{\widetilde{S}^1} \equiv 1.$$

(a) Show that there exists a unique equivalent martingale measure Q^{**} for $\widehat{S}^0$.

(b) Let Q^* be the unique equivalent martingale measure for $S^1 = \widetilde{S}^1/\widetilde{S}^0$. Show that the density of Q^{**} with respect to Q^* on $\mathcal{F}_T$ is given by

$$\frac{\mathrm{d}Q^{**}}{\mathrm{d}Q^*} = S^1_T.$$

(c) Show that for an *undiscounted* payoff $\widetilde{H} \in L^0_+(\mathcal{F}_T)$, we have

$$\widetilde{S}^0_k\, E_{Q^*}\left[\frac{\widetilde{H}}{\widetilde{S}^0_T}\,\middle|\,\mathcal{F}_k\right] = \widetilde{S}^1_k\, E_{Q^{**}}\left[\frac{\widetilde{H}}{\widetilde{S}^1_T}\,\middle|\,\mathcal{F}_k\right], \quad k = 0, \ldots, T.$$

This formula shows that the martingale pricing method is invariant under a so-called (particular) *change of numéraire*.
Hint: Use Bayes' formula (Lemma II.3.1) in the lecture notes.