ETH zürich

Mathematical Foundations for Finance

Exercise 7

Martin Stefanik
ETH Zurich

Stochastic Processes

Let's recall some definitions as well as make some adaptations for continuous time.

Definition 1 (Stochastic process)

A (real-valued) *stochastic process* $X = (X_t)_{t \geq 0}$ is any collection of random variables $X_t : \Omega \to \mathbb{R}$ defined on a common probability space $(\Omega, \mathcal{F}, P)$.

Definition 2 (Filtration)

A *filtration* $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ on a measurable space $(\Omega, \mathcal{F})$ is a family of σ-algebras $\mathcal{F}_t \subseteq \mathcal{F}$ which is increasing in the sense that $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \leq t$.

We will tacitly assume that our filtration satisfies so-called *usual conditions* of being *right-continuous* and *P-complete*. What does this mean?

- Right continuity means that

$$\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{\epsilon>0} \mathcal{F}_{t+\epsilon}.$$

- *P*-completeness means that $\mathcal{F}_0$ contains all *P*-nullsets of $\mathcal{F}$.

The last assumption does not appear to be an unreasonable one from the practical perspective. *P*-completeness means that we know what is possible and what isn't from the very beginning, and a large class of processes used in finance (such as Lévy processes for instance) generate a filtration that is right-continuous already after *P*-completion.

What we gain is that all martingales have versions with *càdlàg trajectories*. This path regularity makes it possible to deal with uncountable (time) index sets using our theory built on countable additivity.

Stochastic Processes

There are three useful ways how to look at stochastic processes, one of which will be used to define the concept of a *predictable process* in continuous time:

1. A collection of random variables $X_t : \Omega \to \mathbb{R}$ indexed by time $t \geq 0$.
2. A family of random functions $t \mapsto X_t(\omega)$ on $[0, \infty)$ indexed by $\omega \in \Omega$; we also speak about the *path* or *trajectory* $X_\bullet(\omega)$ for a fixed $\omega \in \Omega$.
3. A mapping $X : \Omega \times [0, \infty) \to \mathbb{R}$, $(\omega, t) \mapsto X_t(\omega)$ on the product space $\overline{\Omega} := \Omega \times [0, \infty)$.

Stochastic Processes

Definition 3 (Adapted process)

A stochastic process $X = (X_t)_{t \geq 0}$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ is *adapted* to $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ if X_t is $\mathcal{F}_t$-measurable for all $t \geq 0$.

Definition 4 (Predictable process)

A stochastic process $X = (X_t)_{t \geq 0}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is called *predictable* if it is measurable with respect to a σ-algebra $\mathcal{P}$ on $\overline{\Omega} := \Omega \times [0, \infty)$ generated by all $\mathbb{F}$-adapted left-continuous processes when viewed as a mapping $X : \overline{\Omega} \to \mathbb{R}$.

What is important for *us* from practical perspective is that all $\mathbb{F}$-adapted continuous processes are predictable.

Definition 5 (Martingale)

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be a filtered probability space. A (real-valued) stochastic process $X = (X_t)_{t\geq 0}$ is called a *martingale* (with respect to $\mathbb{F}$ and P) if

1. X is adapted to $\mathbb{F}$,
2. $X_t \in L^1(P)$ for all $t \geq 0$,
3. X satisfies the *martingale property*, i.e. $E[X_t \mid \mathcal{F}_s] = X_s$ P-a.s. for $s \leq t$.

- The definition is of course completely analogous for time index sets of the form $[0, T]$ with $T \in \mathbb{R}_+$.
- Unlike in discrete time, a definition using the one-step martingale property of the form $E[X_k \mid \mathcal{F}_{k-1}] = X_k$ P-a.s. for all $k = 1, \ldots, T$ does not make sense since there is no such thing as the smallest possible time step.

Definition 6 (Stopping time)

A random variable $\tau : \Omega \to [0, \infty]$ defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is called an $\mathbb{F}$-*stopping time* if $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$.

Definition 7 (Local martingale)

An adapted stochastic process $X = (X_t)_{t \geq 0}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with $X_0 = 0$ is called a *local martingale* (with respect to P and $\mathbb{F}$) if there exists a sequence of stopping times $(\tau_n)_{n \in \mathbb{N}}$ increasing to ∞ such that for each $n \in \mathbb{N}$ the stopped process $X^{\tau_n} = (X_{t \wedge \tau_n})_{t \geq 0}$ is a $(P, \mathbb{F})$-martingale.

Definition 8 (Brownian motion)

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be filtered probability space. A (real-valued) stochastic process $W = (W_t)_{t \geq 0}$ is a *Brownian motion* with respect to P and $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ if it is adapted to $\mathbb{F}$ and satisfies the following properties:

1. $W_0 = 0$ P-a.s.
2. For $s \leq t$ the increment $W_t - W_s$ is independent (under P) of $\mathcal{F}_s$ with a normal distribution $\mathcal{N}(0, t - s)$ (under P).
3. W has continuous trajectories, i.e. for P-almost all $\omega \in \Omega$, the function $t \mapsto W_t(\omega)$ on $[0, \infty)$ is continuous.

Some Properties of Brownian Motion

Let $W = (W_t)_{t \geq 0}$ be a Brownian motion with respect to P and $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$

- $E[W_t] = E[W_t - W_0] = 0.$
- $\mathrm{Var}(W_t) = \mathrm{Var}(W_t - W_0) = t.$
- W has P-a.s. continuous trajectories, but is nowhere differentiable.
- P almost all trajectories of W attain any $a \in \mathbb{R}$ infinitely many times.
- Quadratic variation of W on $[0, t]$ is t, P-a.s.
- W is a martingale; we have that $E[W_t \mid \mathcal{F}_s] = W_s$.
- W has the *Markov property*; we have that

$$E\left[g(W_u; u \geq T) \mid \sigma(W_s; s \leq T)\right] = E\left[g(W_u; u \geq T) \mid \sigma(W_T)\right].$$

Thank you for your attention!