Coordinator
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Mathematical Foundations for Finance

Exercise sheet 8

Please hand in your solutions until Wednesday, November 11, 12:00 via the course homepage.

Exercise 8.1 Let $W = (W_t)_{t \geq 0}$ and $W' = (W'_t)_{t \geq 0}$ be two *independent* Brownian motions (BM) defined on some probability space $(\Omega, \mathcal{F}, P)$ (without filtration). Show that

(a) $W^1 := -W$ is a BM.

(b) $W_t^2 := W_{T+t} - W_T$, $t \geq 0$, is a BM for any $T \in (0, \infty)$.

(c) $W^3 := \alpha W + \sqrt{1 - \alpha^2} W'$ is a BM for any $\alpha \in [0, 1]$.

(d) Show that the independence of W and W' in (c) cannot be omitted, i.e., if W and W' are *not* independent, then W^3 need not be a BM. Give two examples.

Exercise 8.2 Let $W = (W_t)_{t \geq 0}$ be a Brownian motion defined on some sufficiently rich filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$ is a filtration satisfying the usual conditions.

(a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be an arbitrary convex function. Show that if the stochastic process $\big(f(W_t)\big)_{t \geq 0}$ is integrable, then it is a $(P, \mathbb{F})$-submartingale.
Hint: We have done something similar in discrete time.

(b) Given a $(P, \mathbb{F})$-martingale $(M_t)_{t \geq 0}$ and a measurable function $g\colon \mathbb{R}_+ \to \mathbb{R}$, show that the process

$$\big(M_t + g(t)\big)_{t \geq 0}$$

is a $(P, \mathbb{F})$-supermartingale if and only if g is decreasing, and a $(P, \mathbb{F})$-submartingale if and only if g is increasing.

(c) Show that the following stochastic processes are $(P, \mathbb{F})$-submartingales, but not martingales:

(i) W^2,

(ii) $e^{\alpha W}$ for any $\alpha \in \mathbb{R}$.

Hint: Use the result from (a) and (b), respectively.

(d) Show that any $(P, \mathbb{F})$-local martingale which is null at 0 and uniformly bounded from below is a $(P, \mathbb{F})$-supermartingale.
Hint: We have done this in discrete time already.

Exercise 8.3 In this exercise, we are going to allude to the so-called *Girsanov theorem* that we will prove later in the course in a more general setting. We do this by showing a simple special case. Let $W = (W_t)_{t \in [0,T]}$, $T > 0$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$.

(a) Let $M^\alpha = (M_t^\alpha)_{t \in [0,T]}$ be defined by

$$M_t^\alpha := e^{\alpha W_t - \frac{1}{2}\alpha^2 t}.$$

Show that for any $\alpha \in \mathbb{R}$, M^α is the density process of an equivalent probability measure Q^α defined via its Radon–Nikodým derivative with respect to P given by

$$\frac{dQ^\alpha}{dP} := M_T^\alpha.$$

(b) Show that W is a $(Q, \mathbb{F})$-Brownian motion with constant drift $\beta \in \mathbb{R}$. This means that W satisfies all the defining properties of BM except that instead of $W_t - W_s \sim \mathcal{N}(0, t-s)$ for all $0 \leq s \leq t \leq T$, we have that $W_t - W_s \sim \mathcal{N}(\beta(t-s), t-s)$ for some $\beta \in \mathbb{R}$ to be determined, for all $0 \leq s \leq t \leq T$.
Hint 1: Compute the conditional moment-generating function of $W_t - W_s$ under Q^α.
Hint 2: The Bayes formula from Lemma II.3.1 also holds in continuous time.
Hint 3: Recall that if $Z \sim \mathcal{N}(\mu, \sigma^2)$ then $E\left[e^{uZ}\right] = e^{\mu u + 1/2\sigma^2 u^2}$ for all $u \in \mathbb{R}$.
Hint 4: If Y and Y' are random variables with $E\left[e^{uY} \,\middle|\, \mathcal{G}\right] = E\left[e^{uY'}\right] < \infty$ for all $u \in \mathbb{R}$ and some σ-algebra $\mathcal{G}$, then Y is independent of $\mathcal{G}$.