# Non-Life Insurance: Mathematics and Statistics

## Exercise sheet 5

**Exercise 5.1 Large Claims**

In this exercise we are interested in storm and flood events with claim amounts exceeding CHF 50 million as an example of large claims modeling. Assume that for the total yearly claim amount $S$ of storm and flood events with claim amounts exceeding CHF 50 million we model $S \sim \text{CompPoi}(\lambda, G)$, where $\lambda$ is unknown and $G$ is the distribution function of a Pareto distribution with threshold $\theta = 50$ and unknown tail index $\alpha > 0$. Note that we set $\theta = 50$ as we will work in units of CHF 1 million. Moreover, all numbers of claims and claim sizes and, thus, also the total claim amounts are assumed to be independent across different years. During the years $1986 - 2005$ we observed the following 15 storm and flood events with corresponding claim amounts in CHF millions:

| date | amount in millions | date | amount in millions |
|---|---:|---|---:|
| 20.06.1986 | 52.8 | 18.05.1994 | 78.5 |
| 18.08.1986 | 135.2 | 18.02.1999 | 75.3 |
| 18.07.1987 | 55.9 | 12.05.1999 | 178.3 |
| 23.08.1987 | 138.6 | 26.12.1999 | 182.8 |
| 26.02.1990 | 122.9 | 04.07.2000 | 54.4 |
| 21.08.1992 | 55.8 | 13.10.2000 | 365.3 |
| 24.09.1993 | 368.2 | 20.08.2005 | 1'051.1 |
| 08.10.1993 | 83.8 | | |

Table 1: Dates and claim amounts in CHF millions of the 15 storm and flood events observed during the years 1986-2005.

(a) Show that the MLE $\widehat{\alpha}_n^{\text{MLE}}$ of $\alpha$ for $n$ i.i.d. claim sizes $Y_1, \ldots, Y_n \sim \text{Pareto}(\theta, \alpha)$ is given by

$$\widehat{\alpha}_n^{\text{MLE}} = \left( \frac{1}{n} \sum_{i=1}^{n} \log Y_i - \log \theta \right)^{-1}.$$

(b) According to Lemma 3.8 of the lecture notes (version of March 20, 2019),

$$\frac{n-1}{n} \, \widehat{\alpha}_n^{\text{MLE}}$$

is an unbiased version of the MLE. Estimate $\alpha$ using the unbiased version of the MLE for the storm and flood data given in Table 1.

(c) Calculate the MLE of $\lambda$ for the storm and flood data given in Table 1.

(d) Suppose that we introduce a maximal claims cover of CHF $M = 2$ billion per storm and flood event, i.e. the individual claims are given by $\min\{Y_i, M\}$. Using the estimates of $\alpha$ and $\lambda$ found in parts (b) and (c), calculate the estimated expected total yearly claim amount.

(e) Using the estimates of $\alpha$ and $\lambda$ found in parts (b) and (c), calculate the estimated probability that we observe at least one storm and flood event next year which exceeds the level of CHF $M = 2$ billion.

**Exercise 5.2 Claim Size Distributions (R Exercise)**
Write an R code that generates i.i.d. samples of size $n = 10'000$ from each of the following distributions:

- $\Gamma(\gamma, c)$ with shape parameter $\gamma = \frac{1}{4}$ and scale parameter $c = \frac{1}{40'000}$,
- Weibull$(\tau, c)$ with shape parameter $\tau = 0.54$ and scale parameter $c = 0.000175$,
- LN$(\mu, \sigma^2)$ with mean parameter $\mu = \log\left(2000\sqrt{5}\right)$ and variance parameter $\sigma^2 = \log(5)$,
- Pareto$(\theta, \alpha)$ with threshold $\theta = 10'000\frac{\sqrt{5}}{2+\sqrt{5}}$ and tail index $\alpha = 1 + \frac{\sqrt{5}}{2}$.

Note that the parameters are chosen such that the theoretical expectations and standard deviations are approximately equal to 10'000 and 20'000, respectively, for all the distributions listed above. For each of these i.i.d. samples consider

- the density plot (on the log scale),
- the box plot (on the log scale),
- the plot of the empirical distribution function (on the log scale),
- the plot of the empirical loss size index function,
- the empirical log-log plot,
- the plot of the empirical mean excess function.

Comment on your results.

**Exercise 5.3 Hill Estimator (R Exercise)**
Write an R code that samples 300 i.i.d. observations from a Pareto distribution with threshold $\theta = 10$ and tail index $\alpha = 2$. Create a Hill plot and a log-log plot. What do you observe?

**Exercise 5.4 Pareto Distribution**
Suppose the random variable $Y$ follows a Pareto distribution with threshold $\theta > 0$ and tail index $\alpha > 0$.

(a) Show that the survival function of $Y$ is regularly varying at infinity with tail index $\alpha$.

(b) Show that for $\theta \leq u_1 < u_2$ the expected value of $Y$ within the layer $(u_1, u_2]$ is given by

$$\mathbb{E}\left[Y 1_{\{u_1 < Y \leq u_2\}}\right] = \begin{cases} \theta \frac{\alpha}{\alpha-1}\left[\left(\frac{u_1}{\theta}\right)^{-\alpha+1} - \left(\frac{u_2}{\theta}\right)^{-\alpha+1}\right], & \text{if } \alpha \neq 1, \\ \theta \log\left(\frac{u_2}{u_1}\right), & \text{if } \alpha = 1. \end{cases}$$

(c) Show that for $\alpha > 1$ and $y > \theta$ the loss size index function for level $y$ is given by

$$\mathcal{I}[G(y)] = 1 - \left(\frac{y}{\theta}\right)^{-\alpha+1}.$$

(d) Show that for $\alpha > 1$ and $u > \theta$ the mean excess function of $Y$ above $u$ is given by

$$e(u) = \frac{1}{\alpha - 1}\, u.$$