Lecture 10. $g=0$ Gromov-Witten theory

- Properties of $g=0$ GW invariants
- The WDW relations
- Quantum Cohomology of $\mathbb{P}^2$

§1. Some properties of $g=0$ GW invariants.

Let's consider the case when $X = \mathbb{P}^N$.

Ax1 (Dimension constraint) For $\gamma_1, \dots, \gamma_n \in H^*(X)$,

$$\langle \gamma_1, \dots, \gamma_n \rangle^X_{0,d} = 0$$

unless $\sum \deg_{\mathbb{C}}(\gamma_i) = \dim \overline{M}_{0,n}(X,d)$

Ax2 (S_n - invariance) For $\sigma \in S_n$,

$$\langle \gamma_1 \cdots \gamma_n \rangle^X_{0,d} = \langle \sigma(\gamma_1) \cdots \sigma(\gamma_n) \rangle^X_{0,d}$$

($\because$ σ induces an isom $\phi_\sigma : \overline{\mathcal{M}}_{0,n}(X,d) \to \overline{\mathcal{M}}_{0,n}(X,d)$.)

Ax3 (Fundamental class) If $\gamma_{n+1} = 1 \in H^0(X)$,

$$\langle \gamma_1 \cdots \gamma_n, 1 \rangle^X_{0,d} = 0$$

unless $d=0, n=2$. If $d=0, n=2$,

$$\langle \gamma_1, \gamma_2, \gamma_3 \rangle^X_{0,0} = \int_X \gamma_1 \cup \gamma_2 \cup \gamma_3 .$$

proof). For $1 \le i \le n$, consider the following diagram

$$\begin{tikzcd} \overline{\mathcal{M}}_{0,n+1}(X,d) \arrow[r, "ev_i'"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & X \\ \overline{\mathcal{M}}_{0,n}(X,d) \arrow[ur, "ev_i"'] & \end{tikzcd}$$

$$\Rightarrow \langle \gamma_1, \cdots \gamma_n, \gamma_{n+1} \rangle^X_{0,d}$$

$= \text{degree } 0 \text{ part of } \underbrace{\pi_*(ev_1'^*\gamma_1 \cdots ev_n'^*\gamma_n)}$

$$= \pi_*\pi^*(ev_1^*\gamma_1 \cup \cdots \cup ev_n^*\gamma_n)$$

$$= ev_1^*\gamma_1 \cup \cdots \cup ev_n^*\gamma_n \cap \pi_* \mathbb{1} = 0$$

$\pi_* \mathbb{1} = 0$ bc relative dim of π is 1.

Ax4. (Divisor Equation) If $\gamma_{n+1} = c_1(\mathcal{L}) \in H^2(X)$,

$$\langle \gamma_1 \cdots \gamma_n, \gamma_{n+1} \rangle^X_{0,d} = \left(\int_d \gamma_{n+1}\right) \langle \gamma_1 \cdots \gamma_n \rangle^X_{0,d}$$

Let's use the following lemma

Lemma $\pi : X \longrightarrow Y$ flat, surjective morphism btw smooth irred. variety s.t. fibers are nodal curves. Let $\mathcal{L}$ = line bundle on X. For $y \in Y$, let $X_y = \pi^{-1}(y)$ and let $d = \int_{X_y} c_1(\mathcal{L})$. Then

$$\pi_*(c_1(\mathcal{L})) = d \qquad \in H^0(Y)$$

Proof). We know $\pi_*(c_1(\mathcal{L})) = c$ for some $c \in \mathbb{Q}$.
So, it is enough to show $c = d$

$$\begin{tikzcd} X_y \arrow[r, hook, "\iota'"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, "\pi"] \\ y \arrow[r, hook, "\iota"] & Y \end{tikzcd}$$

$$\begin{aligned} c[y] &= c\,\iota^*[Y] \\ &= \iota^* \pi_* c_1(\mathcal{L}) \\ &= \pi'_* \iota'^* c_1(\mathcal{L}) \\ &= \int_{X_y} c_1(\iota'^* \mathcal{L}) = d[y] \end{aligned}$$

$\Rightarrow c = d$ $\qquad\qquad$ ∎

Proof of divisor equation) Apply above lemma to

$$\pi : \overline{M}_{0,n+1}(X,d) \longrightarrow \overline{M}_{0,n}(X,d).$$

Projection formula $\Rightarrow$ it is enough to check

$$\pi_* ev_{n+1}^* c_1(\mathcal{O}(1)) = d.$$

By the lemma, it is enough to check the fiberwise degree over any point $\in \overline{M}_{0,n}(X,d)$. One can use the $[f:(C,p_1 \ldots p_n) \longrightarrow X] \in M_{0,n}(X,d)$, for instance (we did this on Tuesday) $\square$

ASIDE: Generalizations.

(1) Descendent invariants

We can also consider ψ_i classes on $\overline{M}_{0,n}(X,d)$.

$$\begin{array}{c} \overline{M}_{0,n+1}(X,d) \\ p_i \left(\Big\downarrow \pi \right. \\ \overline{M}_{0,n}(X,d) \end{array} \qquad \rightsquigarrow \psi_i := p_i^* c_1(\omega_\pi) \in H^2(\overline{M}_{0,n}(X,d))$$

$$\langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \rangle_{0,d}^X := \int_{[\overline{M}_{0,n}(X,d)]} \psi_1^{a_1} ev_1^*(\gamma_1) \cup \cdots \cup \psi_n^{a_n} ev_n^*(\gamma_n).$$

Ex In general, $\langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_{n-1}}(\gamma_{n-1}) \, \tau_0(1) \rangle$ is nontrivial. This is because $\pi^*\psi_i \neq \psi_i$

Combining Ax3 & string equation, we get

$$\left\langle \prod_{i=1}^{n-1} \tau_{a_i}(\gamma_i) \, \tau_0(1) \right\rangle_{0,d}^{X} = \sum_{i=1}^{n-1} \langle \tau_{a_i-1}(\gamma_i) \rangle_{0,d}^{X} \quad (\tau_{-1} \equiv 0)$$

(2) For arbitrary X, $\overline{M}_{0,n}(X,d)$ is not necessary smooth & irreducible. Even so, there exist a cycle

$$[\overline{M}_{0,n}(X,d)]^{vir} \in H_{2\,vdim}(\overline{M}_{0,n}(X,d)).$$

Due to: [Li - Tian], [Behrend - Fantechi].

We can define $g=0$ GW invariants by integrating coh. classes against $[\cdots]^{vir}$.

(3) If $H^*(X,\mathbb{Q})$ has odd cohomology classes, the order of γ_i becomes important.

§2. The Witten-Dijkgraaf-Verlinde-Verlinde eq.

For any X, we have the "forgetful map"

$$p : \overline{\mathcal{M}}_{g,n}(X,d) \longrightarrow \overline{\mathcal{M}}_{g,n}$$

Idea Suppose we have an interesting relation

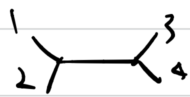
$$\alpha = 0 \qquad \in H^*(\overline{\mathcal{M}}_{g,n})$$

$\Rightarrow$ We can pullback α to $\overline{\mathcal{M}}_{g,n}(X,d)$ & get

$$p^*\alpha \cap [\overline{\mathcal{M}}_{g,n}(X,d)]^{vir} = 0 .$$

This gives a relation among GW invariants. This is in some sense universal (bc it does not depend on X).

▫ WDVV relation.

Let $D(1.2|3.4) = \Big[$  $\Big] \in H^*(\overline{\mathcal{M}}_{0,4})$

$\rightsquigarrow \alpha = D(1.2|3.4) - D(1.3|2.4) = 0 \in H^*(\overline{\mathcal{M}}_{0,4})$

Let $p : \overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, d) \longrightarrow \overline{\mathcal{M}}_{0,4}$.

Consider:

$$\begin{tikzcd} \bigcup D(I_1.d_1 | I_2.d_2) \arrow[r, hook, "J"] \arrow[d] & \overline{M}_{0,n}(\mathbb{P}^2.d) \arrow[d, "p"] \\ D(1.2|3.4) \arrow[r, hook, "\iota"] & \overline{M}_{0.4} \end{tikzcd}$$

Where the sum is over all decomposition

$$I_1 \sqcup I_2 = \{1, \dots n\}, \quad d_1 + d_2 = d$$

Since p is flat,

$$p^* D(1.2|3.4) = \sum J_* [D(I_1.d_1 | I_2.d_2)]$$

Claim: The divisor $D(I_1.d_1 | I_2.d_2)$ has multiplicity 1.

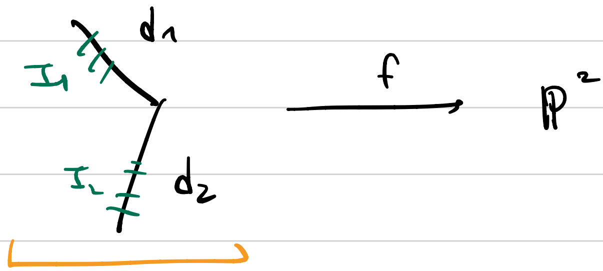
Sketch It is enough to show that $\exists$ open dense substack of $D(I_1.d_1 | I_2 d_2)$ which is reduced.

$\exists$ open substack $U \subset \overline{M}_{0,n}(\mathbb{P}^2.d)$ s.t $\forall_{x \in U}$

$$dp_{[f]}: T_{[f]}U \longrightarrow T_{p[f]} \overline{M}_{0.4}$$

is surjective & $U \cap D(I_1.d_1 | I_2.d_2) \subset D(I_1.d_1 | I_2.d_2)$ is open and dense substack.

e.g



no further degeneration

$\Rightarrow p^{-1}D(1.2|3.4) \cap U$ is smooth hence reduced □

For decorated stable graphs $\Gamma_{I_1.d_1|I_2 d_2}$,

$\Gamma_{I_1 d_1|I_2 d_2} =$ 

& $\xi_\Gamma : \overline{M}_\Gamma(\mathbb{P}^2) \longrightarrow \overline{M}_{0,n}(\mathbb{P}^2,d)$

we have :

$$p^*D(1.2|3.4) = \sum_{\substack{I_1 \cup I_2 = \{1,\dots,n\} \\ d_1, d_2 \geq 0 \\ d_1 + d_2 = d}} \xi_{\Gamma *}\left[\overline{M}_{\Gamma_{I_1 d_1|I_2 d_2}}(\mathbb{P}^2)\right]$$

To understand $[\overline{M}_\Gamma(\mathbb{P}^2)]$, it is useful to have the following lemma

Lemma The evaluation map $ev_i : \overline{M}_{0,n}(\mathbb{P}^2, d) \longrightarrow \mathbb{P}^2$ is flat

Hint : Use generic flatness theorem & $\mathrm{Aut}(\mathbb{P}^2) \curvearrowright \mathbb{P}^2$ transitively. $\square$

$$\begin{tikzcd}
\overline{M}_\Gamma(\mathbb{P}^2) \arrow[r, "j"] \arrow[d] & \overline{M}_{0,I_1+1}(\mathbb{P}^2, d_1) \times \overline{M}_{0,I_2+1}(\mathbb{P}^2, d_2) \arrow[d, "ev_{h_1} \times ev_{h_2}"] \\
\mathbb{P}^2 \arrow[r, "\Delta"] & \mathbb{P}^2 \times \mathbb{P}^2
\end{tikzcd}$$

$$j_* [\overline{M}_\Gamma(\mathbb{P}^2)] = (ev_{h_1} \times ev_{h_2})^* \underbrace{[\Delta_{\mathbb{P}^2}]}_{\text{diagonal of } \Delta_{\mathbb{P}^2}}$$

Suppose $\alpha \in H^*(\overline{M}_{0,I_1+1}(\mathbb{P}^2, d_1) \times \overline{M}_{0,I_2+1}(\mathbb{P}^2, d_2))$.

$$\Rightarrow \int_{[\overline{M}_\Gamma(\mathbb{P}^2)]} j^* \alpha = \int_{[\overline{M}_{0,I_1+1}(\mathbb{P}^2, 1) \times \overline{M}_{0,I_2+1}(\mathbb{P}^2, d_2)]} \alpha \cup (ev_1 \times ev_2)^* (\Delta_{\mathbb{P}^2})$$

$$[\Delta_{\mathbb{P}^2}] = 1 \otimes p + H \otimes H + p \otimes 1 \in H^*(\mathbb{P}^2 \times \mathbb{P}^2).$$

Digression : If $n \geq 2$,

$$\overline{M}_{0,n}(\mathbb{P}^2, d) \xrightarrow{\prod ev_i} (\mathbb{P}^2)^n$$

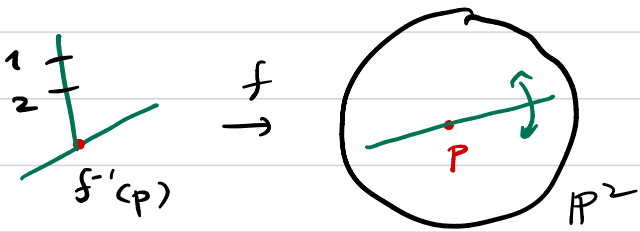
is not flat

Example $n=2$, $d=1$

$$\overline{M}_{0,2}(\mathbb{P}^2, 1) \xrightarrow{ev_1 \times ev_2} \mathbb{P}^2 \times \mathbb{P}^2$$

Over $(p_1, p_2) \notin \Delta_{\mathbb{P}^2}$, $ev_1 \times ev_2$ is bijective.

Over $(p,p) \in \Delta_{\mathbb{P}^2}$, $(ev_1 \times ev_2)^{-1}(p,p) \cong \mathbb{P}^1$

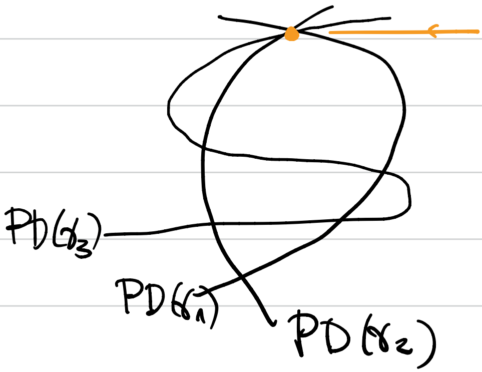


Check $ev_1 \times ev_2$ is a birational map and hence

$$N_1 = \int_{[\overline{M}_{0,2}(\mathbb{P}^2, 1)]} ev_1^*(p) \cup ev_2^*(p) = \int_{\mathbb{P}^2 \times \mathbb{P}^2} p \times p = 1$$

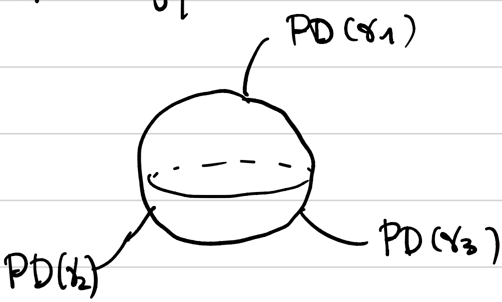
§3 Quantum Cohomology of $\mathbb{P}^2$.

Heuristic $\gamma_1.\gamma_2.\gamma_3 \in H^*(X)$, $\langle \gamma_1.\gamma_2.\gamma_3 \rangle$ computes



physical intersection of the Poincare dual cycles of $\gamma_1.\gamma_2.\gamma_3$

quantum cohomology:



3 cycles meet upto a sphere

$$\langle \gamma_1 * \gamma_2 , \gamma_3 \rangle := \sum_{d=0} \sum_{n=0} \frac{1}{n!} \langle \gamma_1.\gamma_2.\gamma_3 , T^n \rangle_{0,d}^{\mathbb{P}^2}$$

$$= \langle \gamma_1.\gamma_2.\gamma_3 \rangle + \underbrace{\sum_{(d,n)\neq(0,0)} \frac{1}{n!} \langle \gamma_1 \gamma_2 \gamma_3 . T^n \rangle_{0,d}^{\mathbb{P}^2}}_{\text{"meet upto rational curves"}} .$$

$$T = \underset{\substack{\shortparallel \\ T_0}}{t_0 1} + \underset{\substack{\shortparallel \\ T_1}}{t_1 H} + \underset{\substack{\shortparallel \\ T_2}}{t_2 p}$$

Thm $(QH^*(\mathbb{P}^2), *)$ is a commutative associative ring with unit 1.

It is useful to introduce a formal powerseries

$$\Phi(T) = \sum_{n\geq 3} \sum_{d\geq 0} \frac{1}{n!} \langle \tau_0(T)^n \rangle_{0,d}^{\mathbb{P}^2}$$

$$\Phi_{ijk}(T) := \frac{\partial^3}{\partial t_i \partial t_j \partial t_k} \Phi(T) \qquad 0 \leq i, j, k \leq 2$$

The intersection matrix writes : $g_{ij} = \int_{\mathbb{P}^2} T_i \cup T_j$.

and $(g^{ij}) = (g_{ij})^{-1}$.

Def' $T_i * T_j = \sum_{e,f} \Phi_{ije} \, g^{ef} T_f$.

Claim 1 is unit

pf)
$$\Phi_{0jk} = \sum_{n\geq 3} \sum_{d\geq 0} \frac{1}{n!} \langle T_0 T_j T_k T^n \rangle^{\mathbb{P}^2}$$

$$= \int_{\mathbb{P}^2} T_0 \cup T_j \cup T_k \qquad (\because \text{Ax } 3)$$

$$\Rightarrow T_0 * T_j = T_j \quad \forall j \qquad \square$$

<u>Claim</u> $\star$ is associative

Pf).

$$(T_i \star T_j) \star T_k = \sum \Phi_{ije}\, g^{ef}\, T_f \star T_k$$
$$= \sum \Phi_{ije}\, g^{ef}\, \Phi_{fkc}\, g^{cd}\, T_d$$

$$T_i \star (T_j \star T_k) = \sum \Phi_{jke}\, g^{ef}\, T_i \star T_f$$
$$= \sum \Phi_{jke}\, g^{ef}\, \Phi_{ifc}\, g^{cd}\, T_d$$

Since (g^{cd}) is invertible, it is enough to show:

$$\sum \Phi_{ije}\, g^{ef}\, \Phi_{fk\ell} = \sum \Phi_{jke}\, g^{ef}\, \Phi_{if\ell}$$

Let

$$F(i,j \mid k,\ell) = \sum_{e,f} \Phi_{i,j,e}\, g^{ef}\, \Phi_{f,k,\ell}$$
$$= \sum_{\substack{n_1,n_2 \geq 0 \\ d_1,d_2 \geq 0}} \frac{1}{n_1!\, n_2!} \langle T_i T_j T_e T^{n_1} \rangle^{\mathbb{P}^2}_{0,d_1}\, g\, \langle T_f T_k T_\ell T^{n_2} \rangle^{\mathbb{P}^2}_{0,d_2}$$

Then the associativity is equivalent to

$$F(i,j \mid k,\ell) = F(j,k \mid i,\ell)$$

<u>Ex</u> Prove the above equality. $\square$