Lecture 12. DR formula, Localization

- Pixton's formula for DR cycles
- Atiyah-Bott localization
- T-fixed locus of $\overline{M}_{g,n}(\mathbb{P}^1,d)$

§1. Pixton's formula for DR cycles

$A=(a_1,\cdots,a_n)\in\mathbb{Z}^n$. s.t $\sum_i a_i=0$.

$r>0$: positive integer.

$\Gamma\in G_{g,n}$: stable graph

Def A weighting mod r of Γ is a function

$$w: H(\Gamma)\longrightarrow\{0,1,\cdots,r-1\} \quad \text{s.t}$$

$\uparrow$ set of half edges

(I) $i\in$ Markings $\quad w(i)\equiv a_i \pmod r$

(II) $e=(h,h')\in$ Edges, $\quad w(h)+w(h')\equiv 0 \pmod r$

(III) $v\in$ Vertex, $\quad \sum_{h\vdash v} w(h)\equiv 0 \pmod r$

$\rightsquigarrow$ $W_{\Gamma,r}$:= set of weightings mod r on Γ. $|W_{\Gamma,r}|=r^{h^1(\Gamma)}$.

Def (Pixton) $P_g^{d,r}(A)$ is the degree d component of

$$\sum_{\Gamma\in G_{g,n}}\sum_{w\in W_{\Gamma,r}}\frac{1}{|\mathrm{Aut}\,\Gamma|}\frac{1}{r^{h^1(\Gamma)}}\cdot$$

$$\xi_{\Gamma *}\left[\prod_{i=1}^{n}\exp\left(\tfrac{1}{2}a_i^2\psi_i\right)\cdot\prod_{e=(h,h')}\frac{1-\exp\left(-\frac{w(h)w(h')}{2}(\psi_h+\psi_{h'})\right)}{\psi_h+\psi_{h'}}\right]$$

$\uparrow$ tautological!

↳ $P_g^{d,r}(A)$ is polynomial in r $(\gg 0)$.

$P_g^{d}(A)$ = const. term of $P_g^{d,r}(A)$

$C = C_1 \cup C_2 \qquad \mathrm{Pic}^0(C) \cong \mathrm{Pic}^0(C_1) \times \mathrm{Pic}^0(C_2)$

Digression (DR cycles on compact type curves)

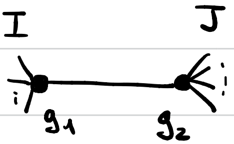
$$\mathcal{M}^{ct}_{g,n} = \{ (C, p_i) \mid \mathrm{Pic}^0(C) \text{ is compact} \} \xhookrightarrow{\text{open}} \overline{\mathcal{M}}_{g,n}$$

$\Leftrightarrow$ the dual graph of C has no loops.

Over $\mathcal{M}^{ct}_{g,n}$, Abel-Jacobi section naturally extends.

Let

$$\Theta = \sum_{i=1}^{n} \frac{1}{2} a_i^2 \psi_i + \sum_{\substack{I \cup J = [n] \\ g_1 + g_2 = g}} - \frac{a_I^2}{2} \left[\qquad \right]$$



$\in H^2(\overline{\mathcal{M}}_{g,n})$

$a_I = - a_J$

where $a_I = \sum_{i \in I} a_i$.

Thm (Hain)

$$DR^{ct}_g(A) := DR_g(A)|_{\mathcal{M}^{ct}_{g,n}} = [\exp(\Theta)]_{\deg = g}$$

Exercise $P_g^g(A) | \mathcal{M}^{ct}_{g,n} = [\exp \Theta]_{\deg = g}$

(Hint : use self-intersection formula for boundary divisors)

□ Formula for λ_g.

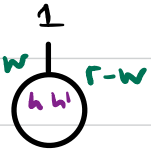
Recall $DR_g(\phi) = (-1)^g \lambda_g \in H^{2g}(\overline{\mathcal{M}}_g)$.

Exercise from middle school:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}, \quad \sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

$g=1$: λ_1 on $\overline{\mathcal{M}}_{1,1}$ $\quad \boxed{a_1 = 0}$

$\Gamma =$ 

$|\mathrm{Aut}\,\Gamma| = 2$.

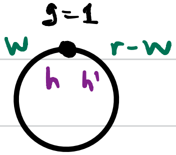
$$P_1^{1,r}(0) = \frac{1}{2}\frac{1}{r}\xi_{\Gamma *}\left[\frac{1-\exp\left(-\frac{w(h)w(h')}{2}(\psi_h+\psi_{h'})\right)}{\psi_h+\psi_{h'}}\right]_{\deg=0}$$

$$\rightsquigarrow \frac{1}{2}\sum_{w=0}^{r-1} w(r-w) = \frac{1}{2}\sum_{w=0}^{r-1} -w^2 + \underbrace{r\cdot\sum_{w=0}^{r-1} w}_{\text{higher order term}}$$

$$= -\frac{1}{2}\cdot\frac{1}{6}r + O(r^2)$$

$$\Rightarrow \lambda_1 = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{6}[\Gamma] = \frac{1}{24}\xi_{\Gamma *}1$$

$g=2$.

$\Gamma_1 =$ 

$|\mathrm{Aut}\,\Gamma_1| = 2$

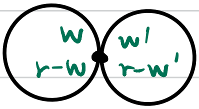
$$\frac{1-\exp\left(-\frac{w(h)w(h')}{2}(\psi_h+\psi_{h'})\right)}{\psi_h+\psi_{h'}} = -\frac{1}{8}(w(h)w(h'))^2(\psi_h+\psi_{h'})$$

$$\rightsquigarrow \sum_{w=0}^{r-1}(w(r-w))^2 = \sum_{w=1}^{r-1} w^4 \underbrace{-2r\sum_{w=1}^{r-1} w^3 + r^2\sum_{w=1}^{r-1} w^2}_{\text{higher order terms}}$$

$$= -\frac{1}{30}r + o(r^2)$$

Contribution of $\Gamma_1 = \underbrace{\frac{1}{2}}_{\text{aut.}}\left(-\frac{1}{8}\right)\left(-\frac{1}{30}\right)\cdot 2\ \xi_{\Gamma*}(\psi_h)$

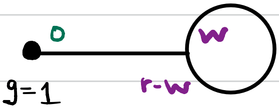
.

$\Gamma_2 =$ 

$|\mathrm{Aut}\,\Gamma_2| = 8$

$$\left(\frac{1}{2}\sum_{w=0}^{r-1} w(r-w)\right)\left(\frac{1}{2}\sum_{w'=0}^{r-1} w'(r-w')\right) = \frac{1}{12}\cdot\frac{1}{12}\ r^2 + o(r^3)$$

Contribution of $\Gamma_2 = \frac{1}{8}\cdot\frac{1}{12}\cdot\frac{1}{12} = \frac{1}{1152}\ \xi_{\Gamma_2*}1$

$\Gamma_3 =$ 

← does not contribute!

$$\Rightarrow \lambda_2 = \frac{1}{240} \xi_{\Gamma_1 *} \psi_h + \frac{1}{1152} \xi_{\Gamma_2 *} 1 .$$

Ex A graph $\Gamma \in G_g$ which has a separating edge does not appear in $P_g^g(\phi)$.

How Bernoulli numbers appear?

$$\frac{t}{e^t - 1} = \sum_{j=0}^{\infty} B_j \frac{t^j}{j!} \quad \text{(Taylor expansion at } t=0\text{)}$$

Check (Faulhaber formula) Let p : positive integer

$$\sum_{k=1}^{n} k^p = \sum_{k=0}^{p} \frac{(-1)^{p-k}}{k+1} \binom{p}{k} B_{p-k} n^{k+1} .$$

appears when we compute weightings of edges of Γ.

We will see Bernoulli numbers in other context (Chiodo's formula)

§2. Atiyah - Bott localization formula

M : nonsingular algebraic variety / $\mathbb{C}$

$T = (\mathbb{C}^*)^r \curvearrowright M$. $\quad H_T^*(pt) = \mathbb{Q}[t_1, \cdots t_r]$

$$\begin{tikzcd} M^T = \{ x \in M \mid t \cdot x = x \ \forall t \in T \} \arrow[r, hook, "i", "\text{closed}"'] & M \end{tikzcd}$$

$$\shortparallel$$

$\bigsqcup_j M_j^T \leftarrow$ connected components

Atiyah - Bott localization has two parts.

$i : M^T \longrightarrow M$ is T-equivariant and hence induces a map

$$i_* : H_T^*(M^T) \longrightarrow H_T^*(M)$$

Thm 1 i_* is an isomorphism after inverting t_i

pf) Let's try to prove this for Chow groups.

Let $U = M \setminus M^T$. We have an exact sequence

$$CH_*^T(U,1) \to CH_*^T(M^T) \xrightarrow{\boxed{i_*}} CH_*^T(M) \longrightarrow CH_*^T(U) \to 0$$

$\uparrow$ T- equiv. 1^{st} higher Chow group. $\qquad\qquad \uparrow$ T - equiv. Chow group

Since U is the complement of T-fixed points, $[U/T]$ is DM-stack (i.e stabilizer group is finite)

$\Rightarrow$ $CH^T_*(U) = CH_*([U/T]) = 0$ $\quad * < 0$

$CH_*(U,1) = 0$ $\quad * < -1$.

ι_* is $\mathbb{Q}[t]$-module homomorphism

$\Rightarrow$ $\ker \iota_*$ & $\operatorname{coker} \iota_*$ is killed by multiplying t^N, $N \gg 0$. ☒

Rmk We haven't used the fact that M is nonsingular.

$\Rightarrow$ Thm 1 holds for any M (at least when M = DM stack)

⚠ When M is a DM-stack $T \circlearrowright M^T$ can be nontrivial and the proof needs a slight modification.

The second part of the localization theorem describes i_*.

- Equivariant euler class

Def T-equivariant vector bundle is a vector bundle $p : V \longrightarrow M$ with T-action on V s.t. $\forall x \in M.\ t \in T,$

$$t : V_x \longrightarrow V_{t \cdot x} \quad \text{is linear isom.}$$

$\Rightarrow$

$$\begin{tikzcd} V \times_T ET \arrow[d] \\ M \times_T ET \end{tikzcd}$$

$\longleftarrow$ vector bundle of rank $= \mathrm{rk}\, V$

Def $e_T(V) := e(V \times_T ET) \in H^*(M \times_T ET) = H_T^*(M)$

Example $T = \mathbb{C}^* \curvearrowright pt$. Then

$$\{T\text{-equiv. vbdl on pt}\} \longleftrightarrow \left\{ \begin{array}{l} \text{finite dim'l} \\ T\text{-representation} \end{array} \right\}$$

Let $V = \mathbb{C}$. st $t.v = t^n v$ $\quad t \in \mathbb{C}^*$. Then

$$e_T(V) = -nt \quad \in H_T^*(pt) = \mathbb{Q}[t]$$

(Hint: we appoximate $ET = \mathbb{C}^{N+1} - \{0\}$. Then $V \times_T ET$ is a line bundle $\mathcal{O}_{\mathbb{P}^N}(-n)$).

On M_j^T,

$$0 \to \overset{wt=0}{TM_j^T} \to TM|_{M_j^T} \to \overset{wt\neq 0}{Nor_j} \to 0$$

T-equiv. vbdl on M_j^T.

Thm 2 $$[M] = \sum_j \iota_{j*} \frac{[M_j]}{e_T(Nor_j)} \quad \text{in } H_T^*(M)(t_1,\cdots,t_n)$$

pf) Proof is a simple consequence of excess intersection formula.

Recall $V \xhookrightarrow{\iota} W$, both V, W are smooth. Then

$$\iota^*\iota_*\alpha = e(N_V W)\cap\alpha \qquad \forall \alpha \in H^*(V).$$

↳ also holds in T-equiv setting.

By **Thm 1**

$$[M] = \sum_{j=1}^{m} \iota_{j*}\alpha_j \qquad \exists\, \alpha_j \in H_T^*(M_j^T)(t_1,\cdots,t_r)$$

Excess intersection ⇒

$$\begin{cases} \underset{\substack{\| \\ [M_j]}}{\iota_j^*\iota_{j*}\alpha_j} = e_T(Nor_j)\cap\alpha_j \\ \iota_k^*\iota_{j*} = 0 \quad \text{if } k\neq j \end{cases}$$

$e_T(Nor_j)$ is invertible in $H_T^*(M_j^T) \otimes \mathbb{Q}(t_1, \dots t_r) \Rightarrow \alpha_j = \dfrac{[M_j]}{e_T(Nor_j)}$

▨

- How to use it practically?

M = smooth, proper scheme / $\mathbb{C}$ $\circlearrowleft T$.

$\gamma \in H^*(M)$

Goal: Compute the integral $\int_M \gamma \in \mathbb{Q}$

Lemma Consider the restriction $H_T^*(M) \longrightarrow H^*(M)$ (setting $t_i = 0$). Choose a lift $\tilde{\gamma} \in H_T^*(M)$. Then

$$\underbrace{\int_M}_{\text{usual pushforward}} \gamma = \underbrace{\int_M}_{\text{equivariant pushforward}} \tilde{\gamma} \quad \in \mathbb{Q}$$

proof

$$\begin{tikzcd} M \arrow[r, hook, "\rho'"] \arrow[d, "\varepsilon_0"'] & M \times_T ET \arrow[d, "\textcircled{ε}"] \\ * \arrow[r, hook, "\rho"] & ET \end{tikzcd}$$

flat

$\varepsilon_{0*}\rho'^{*}\hat{\gamma} = \rho^{*}\varepsilon_{*}\tilde{\gamma}$

▨

Cor $$\int_M \gamma = \sum_{j=1}^{m} \int_{M_j^T} \frac{\tilde{\gamma}|_{M_j^T}}{e_T(M_j^T, M)} \in \mathbb{Q}$$

pf) Lemma + Thm 2. ☒

Example $M = \mathbb{P}^1$, $H^*(M) = \mathbb{Q}[H]/H^2$, $\gamma = H \in H^*(M)$.

$T = \mathbb{C}^* \circlearrowright M$ by $t.[z_0 : z_1] = [z_0 : t z_1]$.

Two fixed pts : $[0]$ & $[\infty]$.

$H_T^*(\mathbb{P}^1) = \mathbb{Q}[H.t]/H(H-t)$. $\tilde{\gamma} = H + at$. $a \in \mathbb{Q}$

$$\int_{\mathbb{P}^1} \gamma = \int_{\mathbb{P}^1} \tilde{\gamma} = \frac{t+at}{t} + \frac{at}{-t} = 1$$

$$\begin{tikzcd} \mathbb{P}^1 \times_T ET \arrow[d] & \approx & \mathbb{P}(\mathcal{O}\oplus\mathcal{O}(1)) \arrow[d] & 0 \circlearrowright \ \circlearrowright \infty \\ BT & & \mathbb{CP}^\infty & \end{tikzcd}$$

$$H = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(1))}(\overset{\substack{\text{T-inv. divisor} \\ \swarrow}}{D_\infty}))$$

§3. Fixed locus of $\overline{M}_{g,n}(\mathbb{P}^1, d) \circlearrowleft T$.

Let $T \circlearrowright \mathbb{P}^1$ (say, $t[z_0 : z_1] = [z_0 : tz_1]$) with two fixed pts $[0]$ & $[\infty] \in \mathbb{P}^1$.

$$\overline{M}_{g,n}(\mathbb{P}^1, d) \ni [f : C \longrightarrow \mathbb{P}^1],$$

$$t.[f] = [C \xrightarrow{f} \mathbb{P}^1 \xrightarrow{t} \mathbb{P}^1]$$

Q What is the T-fixed locus?

If $f : C \longrightarrow \mathbb{P}^1$ factors through $[0]$ or $[\infty]$, it is obviously T-fixed, i.e. $t.[f] = [f]$

T-fixed point for DM-stack is a bit more subtle!

Def $[f] \in \overline{M}_{g,n}(\mathbb{P}^1, d)$ is a T-fixed locus if for any $t \in T$, $\exists\ \phi_t \in \mathrm{Aut}(C, p_i)$ s.t

$$\begin{tikzcd}
C \arrow[r, "t.f"] \arrow[d, "\phi_t"', "\cong"] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{P}^1 \\
C \arrow[ur, "f"'] &
\end{tikzcd}$$

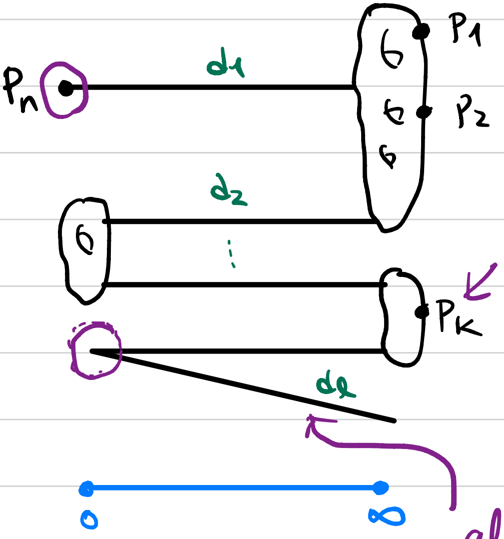
Example $[f_d : \overset{C}{\overset{\|}{\mathbb{P}^1}} \xrightarrow{z \mapsto z^d} \mathbb{P}^1] \in \overline{M}_{0,0}(\mathbb{P}^1, d)$.

Then $[f_d]$ is T-fixed locus.

$$\begin{tikzcd}
C \arrow[r, "f_d"] \arrow[d, "\cong"', "\phi_t = \sqrt[d]{t}\cdot"'] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{P}^1 \arrow[d, "t"] \\
C \arrow[r, "f_d"] & \mathbb{P}^1
\end{tikzcd}$$

(In fact $T \circlearrowright M^T$ nontrivially when $d \geq 2$)

In general: T-fixed locus of $\overline{M}_{g,n}(\mathbb{P}^1, d)$.



※ special points (markings & ramification pts) should be map to T-fixed points of the target

contracted components should be stable

always of the form $z \mapsto z^{d_e}$